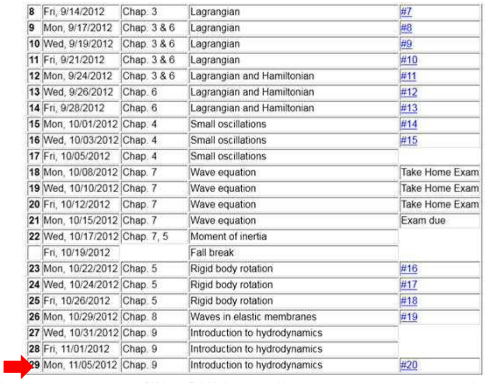
PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 29:

Introduction to hydrodynamics

1. Euler's equation for fluid dynamics
2. Potential flow
3. Bernoulli's integrals



8	Fri, 9/14/2012	Chap. 3	Lagrangian	#7
9	Mon, 9/17/2012	Chap. 3 & 6	Lagrangian	#8
10	Wed, 9/19/2012	Chap. 3 & 6	Lagrangian	#9
11	Fri, 9/21/2012	Chap. 3 & 6	Lagrangian	#10
12	Mon, 9/24/2012	Chap. 3 & 6	Lagrangian and Hamiltonian	#11
13	Wed, 9/26/2012	Chap. 6	Lagrangian and Hamiltonian	#12
14	Fri, 9/28/2012	Chap. 6	Lagrangian and Hamiltonian	#13
15	Mon, 10/01/2012	Chap. 4	Small oscillations	#14
16	Wed, 10/03/2012	Chap. 4	Small oscillations	#15
17	Fri, 10/05/2012	Chap. 4	Small oscillations	
18	Mon, 10/08/2012	Chap. 7	Wave equation	Take Home Exam
19	Wed, 10/10/2012	Chap. 7	Wave equation	Take Home Exam
20	Fri, 10/12/2012	Chap. 7	Wave equation	Take Home Exam
21	Mon, 10/15/2012	Chap. 7	Wave equation	Exam due
22	Wed, 10/17/2012	Chap. 7, 5	Moment of inertia	
	Fri, 10/19/2012		Fall break	
23	Mon, 10/22/2012	Chap. 5	Rigid body rotation	#16
24	Wed, 10/24/2012	Chap. 5	Rigid body rotation	#17
25	Fri, 10/26/2012	Chap. 5	Rigid body rotation	#18
26	Mon, 10/29/2012	Chap. 8	Waves in elastic membranes	#19
27	Wed, 10/31/2012	Chap. 9	Introduction to hydrodynamics	
28	Fri, 11/01/2012	Chap. 9	Introduction to hydrodynamics	
29	Mon, 11/05/2012	Chap. 9	Introduction to hydrodynamics	#20

Some details on the velocity potential

Continuity equation :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \nabla \rho \cdot \mathbf{v} = 0$$

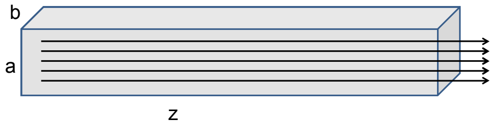
For incompressible fluid : $\rho = (\text{constant})$

$\Rightarrow \nabla \cdot \mathbf{v} = 0$

Irrotational flow : $\nabla \times \mathbf{v} = 0 \qquad \Rightarrow \mathbf{v} = -\nabla \Phi$

$\Rightarrow \nabla^2 \Phi = 0$

Example – uniform flow



$$\nabla^2\Phi = 0$$

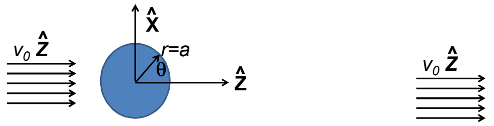
$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

Possible solution :

$$\Phi = -v_o z$$

$$\mathbf{v} = -\nabla\Phi = v_o\hat{\mathbf{z}}$$

Example – flow around a long cylinder (oriented in the ***Y*** direction)



$$\nabla^2\Phi = 0$$

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 0$$

Laplace equation in cylindrical coordinates

($r,\theta,$defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{\partial^2\Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r,\theta,y)=\Phi(r,\theta)$$

From boundary condition : $v_z(r\to\infty)=v_0$

$$\frac{\partial\Phi}{\partial z}(r\to\infty)=-v_0 \qquad \Rightarrow \Phi(r\to\infty,\theta)=-v_0 r\cos\theta$$

Note that : $\frac{\partial^2\cos\theta}{\partial\theta^2}=-\cos\theta$

Guess form : $\Phi(r,\theta)=f(r)\cos\theta$

Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial f}{\partial r}-\frac{1}{r^2}f=0$$

$$f(r)=Ar+\frac{B}{r} \qquad \text{where } A,B \text{ are constants}$$

Boundary condition on cylinder surface :

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=a}=0$$

$$\frac{df}{dr}(r=a)=0=A-\frac{B}{a^2}$$

$$\Rightarrow B=Aa^2$$

Boundary condition at ∞ : $\Rightarrow A=-v_0$

$$\Phi(r,\theta)=-v_0\left(r+\frac{a^2}{r}\right)\cos\theta$$

$$v_r=-\frac{\partial\Phi}{\partial r}=v_0\left(1-\frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta=-\frac{1}{r}\frac{\partial\Phi}{\partial\theta}=v_0\left(1+\frac{a^2}{r^2}\right)\sin\theta$$

For homework; consider similar boundary value problem for a spherical obstruction

Laplacian in spherical polar coordinates :

$$\nabla^2\Phi=0=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Momentum in fluids:

Newton - Euler equation of motion :

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}_{applied}-\frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0$

Want to find an expression for the momentum density : $\rho\mathbf{v}$

$$\rho\frac{\partial\mathbf{v}}{\partial t}+\rho(\mathbf{v}\cdot\nabla)\mathbf{v}=\rho\mathbf{f}_{applied}-\nabla p$$

$$\frac{\partial(\rho\mathbf{v})}{\partial t}-\mathbf{v}\frac{\partial\rho}{\partial t}+\rho(\mathbf{v}\cdot\nabla)\mathbf{v}=\rho\mathbf{f}_{applied}-\nabla p$$

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\mathbf{v}\nabla\cdot(\rho\mathbf{v})+\rho(\mathbf{v}\cdot\nabla)\mathbf{v}=\rho\mathbf{f}_{applied}-\nabla p$$

Momentum in fluids -- continued

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\mathbf{v}\nabla\cdot(\rho\mathbf{v})+\rho(\mathbf{v}\cdot\nabla)\mathbf{v}=\rho\mathbf{f}_{applied}-\nabla p$$

Defining the "stress" tensor :

$$T_{ij}=p\delta_{ij}+\rho v_i v_j$$

The momentum equation can be written in component form :

$$\frac{\partial(\rho v_i)}{\partial t}+\sum_{j=1}^{3}\frac{\partial T_{ij}}{\partial x_j}=\rho f_i$$

Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial\mathbf{v}}{\partial t}+\nabla\left(\tfrac{1}{2}v^2\right)-\mathbf{v}\times(\nabla\times\mathbf{v})=\mathbf{f}_{applied}-\frac{\nabla p}{\rho}$$

Consider the following restrictions :

1. $(\nabla\times\mathbf{v})=0$ "irrotational flow"

   $\Rightarrow \mathbf{v}=-\nabla\Phi$

2. $\mathbf{f}_{applied}=-\nabla U$ conservative applied force
3. $\rho \neq$ (constant) isentropic fluid

A little thermodynamics

First law of thermodynamics : $dE_{\text{int}}=dQ-dW$

For isentropic conditions : $dQ=0$

$$dE_{\text{int}}=-dW=-pdV$$

Solution of Euler's equation for fluids – isentropic (continued)

$$dE_{\text{int}}=-dW=-pdV$$

In terms of mass density : $\rho=\frac{M}{V}$

For fixed M and variable V : $d\rho=-\frac{M}{V^2}dV$

$$dV=-\frac{M}{\rho^2}d\rho$$

In terms in intensive variables : Let $E_{\text{int}}=M\varepsilon$

$$dE_{\text{int}}=Md\varepsilon=-dW=-pdV=M\frac{p}{\rho^2}d\rho$$

$$d\varepsilon=\frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{dQ=0}=\frac{p}{\rho^2}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\left(\frac{\partial\varepsilon}{\partial\rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

Consider : $\nabla\varepsilon = \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{dQ=0} \nabla\rho = \frac{p}{\rho^2}\nabla\rho$

Rearranging : $\nabla\left(\varepsilon + \frac{p}{\rho}\right) = \frac{\nabla p}{\rho}$

Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial\mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

$$\frac{\nabla p}{\rho} = \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$\nabla\times\mathbf{v} = 0$ $\qquad$ $\mathbf{v} = -\nabla\Phi$ $\qquad$ $\mathbf{f}_{applied} = -\nabla U$

$$\frac{\partial\left(-\nabla\Phi\right)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Energy in isentropic fluids

Newton - Euler equation of motion :

$$\frac{\partial\mathbf{v}}{\partial t} + \left(\mathbf{v}\cdot\nabla\right)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\mathbf{v}\right) = 0$

Want to find an expression for the energy density : $\rho\varepsilon + \tfrac{1}{2}\rho v^2$

where $\tfrac{1}{2}\rho v^2$ represents kinetic energy density of center of mass

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2\right) = \tfrac{1}{2}\frac{\partial\rho}{\partial t}v^2 + \rho\mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial t} = -\nabla\cdot\left(\tfrac{1}{2}\rho v^2\mathbf{v}\right) - \mathbf{v}\cdot\nabla p + \rho\mathbf{v}\cdot\mathbf{f}_{applied}$$

Energy in isentropic fluids -- continued

$$\text{Recall: } \nabla\left(\varepsilon+\frac{p}{\rho}\right)=\frac{\nabla p}{\rho}$$

$$\frac{\partial(\rho\varepsilon)}{\partial t}=-\nabla\cdot\left[(\rho\varepsilon+p)\mathbf{v}\right]+\mathbf{v}\cdot\nabla p$$

Combining internal energy and center of mass energy:

$$\frac{\partial\left(\rho\varepsilon+\frac{1}{2}\rho v^2\right)}{\partial t}=-\nabla\cdot\left[\left(\rho\varepsilon+p+\frac{1}{2}\rho v^2\right)\mathbf{v}\right]+\rho\mathbf{v}\cdot\mathbf{f}_{applied}$$

Application of fluid equations to the case of air in equilibrium plus small perturbation

Newton - Euler equation of motion:

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}_{applied}-\frac{\nabla p}{\rho}$$

Continuity equation: $\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0$

Near equilibrium:

$$\rho=\rho_0+\delta\rho$$

$$p=p_0+\delta p$$

$$\mathbf{v}=0+\delta\mathbf{v}$$

$$\mathbf{f}_{applied}=0$$

Equations to lowest order in perturbation:

$$\frac{\partial\delta\mathbf{v}}{\partial t}=-\frac{\nabla\delta p}{\rho_0}$$

$$\frac{\partial\delta\rho}{\partial t}+\rho_0\nabla\cdot\delta\mathbf{v}=0$$