PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 32:

Wave equation for sound

1. **Green's function for wave equation**

17	Fri, 10/05/2012	Chap. 4	Small oscillations	
18	Mon, 10/08/2012	Chap. 7	Wave equation	Take Home Exam
19	Wed, 10/10/2012	Chap. 7	Wave equation	Take Home Exam
20	Fri, 10/12/2012	Chap. 7	Wave equation	Take Home Exam
21	Mon, 10/15/2012	Chap. 7	Wave equation	Exam due
22	Wed, 10/17/2012	Chap. 7, 5	Moment of inertia	
	Fri, 10/19/2012		Fall break	
23	Mon, 10/22/2012	Chap. 5	Rigid body rotation	#16
24	Wed, 10/24/2012	Chap. 5	Rigid body rotation	#17
25	Fri, 10/26/2012	Chap. 5	Rigid body rotation	#18
26	Mon, 10/29/2012	Chap. 8	Waves in elastic membranes	#19
27	Wed, 10/31/2012	Chap. 9	Introduction to hydrodynamics	
28	Fri, 11/01/2012	Chap. 9	Introduction to hydrodynamics	
29	Mon, 11/05/2012	Chap. 9	Introduction to hydrodynamics	#20
30	Wed, 11/07/2012	Chap. 9	Sound waves	
31	Fri, 11/09/2012	Chap. 9	Linear sound waves	#21
32	Mon, 11/12/2012	Chap. 9	Green's function for linear sound	
33	Wed, 11/14/2012	Chap. 9	Non-linear sound	
34	Fri, 11/16/2012	Chap. 9	Non-linear sound	Take Home Exam
35	Mon, 11/19/2012	Chap. 10	Surface waves	Take Home Exam
	Wed, 11/21/2012		*Thanksgiving Holiday*	
	Fri, 11/23/2012		*Thanksgiving Holiday*	
36	Mon, 11/26/2012	Chap. 10	Surface waves	Exam due

Other solutions to wave equation:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

Plane wave solution :

$$\Phi(\mathbf{r},t) = Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Solution in terms of Green's function :

$$\Phi(\mathbf{r},t) = \Phi_0(\mathbf{r},t) + \int d^3r' \int dt'\, G(\mathbf{r}-\mathbf{r}', t-t')\, f(\mathbf{r}',t')$$

where

$\Phi_0(\mathbf{r},t)$ is a solution to the homogeneous equation, and

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Wave equation with source -- continued:

We can show that :

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{\delta\left(t' - \left(t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Derivation of Green's function for wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Recall that

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')}d\omega$$

Derivation of Green's function for wave equation -- continued

$$\text{Define:}\quad \tilde{G}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} G(\mathbf{r},t)e^{i\omega t}dt$$

$$G(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(\mathbf{r},\omega)e^{-i\omega t}d\omega$$

$\tilde{G}(\mathbf{r},\omega)$ must satisfy:

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}') \quad \text{where } k^2 = \frac{\omega^2}{c^2}$$

Derivation of Green's function for wave equation -- continued

$$\left(\nabla^2 + k^2\right)\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

Solution assuming isotropy in $\mathbf{r}-\mathbf{r}'$:

$$\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega) = \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Check -- Define $R \equiv |\mathbf{r}-\mathbf{r}'|$ and for $R > 0$:

$$\left(\nabla^2 + k^2\right)\widetilde{G}(R,\omega) = \frac{1}{R}\frac{d^2}{dR^2}\left(R\widetilde{G}(R,\omega)\right) + k^2\widetilde{G}(R,\omega) = 0$$

Derivation of Green's function for wave equation -- continued

For $R > 0$:

$$\left(\nabla^2 + k^2\right)\widetilde{G}(R,\omega) = \frac{1}{R}\frac{d^2}{dR^2}\left(R\widetilde{G}(R,\omega)\right) + k^2\widetilde{G}(R,\omega) = 0$$

$$\frac{d^2}{dR^2}\left(R\widetilde{G}(R,\omega)\right) + k^2\left(R\widetilde{G}(R,\omega)\right) = 0$$

$$\left(R\widetilde{G}(R,\omega)\right) = A\,e^{ikR} + Be^{-ikR}$$

$$\Rightarrow\ \widetilde{G}(R,\omega) = A\frac{e^{ikR}}{R} + B\frac{e^{-ikR}}{R}$$

Derivation of Green's function for wave equation – continued
need to find *A* and *B*.

$$\text{Note that :}\quad \nabla^2 \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = -\delta(\mathbf{r}-\mathbf{r}')$$

$$\Rightarrow A = B = \frac{1}{4\pi}$$

$$\boxed{\widetilde{G}(R,\omega) = \frac{e^{\pm ikR}}{4\pi R}}$$

Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{G}(\mathbf{r}-\mathbf{r}',\omega) e^{-i\omega(t-t')} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} e^{-i\omega(t-t')} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} e^{-i\omega(t-t')} d\omega$$

Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} e^{-i\omega(t-t')} d\omega$$

Noting that $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega u} d\omega \;=\; \delta(u)$

$$\Rightarrow G(\mathbf{r}-\mathbf{r}', t-t') = \frac{\delta\left(t-\left(t'\mp\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

For time harmonic forcing term we can use the corresponding Green's function:

$$\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

In fact, this Green's function is appropriate for boundary conditions at infinity. For surface boundary conditions where we know the boundary values or their gradients, the Green's function must be modified.

Green's theorem

Consider two functions $h(\mathbf{r})$ and $g(\mathbf{r})$

Note that : $\int_V \left(h\nabla^2 g - g\nabla^2 h\right) d^3r = \oint_S \left(h\nabla g - g\nabla h\right)\cdot \hat{\mathbf{n}} d^2 r$

$$\nabla^2\tilde{\Phi} + k^2\tilde{\Phi} = -\tilde{f}(\mathbf{r},\omega)$$

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

$$h \leftrightarrow \tilde{\Phi}; \qquad g \leftrightarrow \tilde{G}$$

$$\int_V \left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right) f\left(\mathbf{r},\omega\right)\right) d^3r =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right) - \tilde{G}\left(|\mathbf{r}-\mathbf{r}'|,\omega\right)\nabla\tilde{\Phi}(\mathbf{r},\omega)\right)\cdot \hat{\mathbf{n}} d^2 r$$

$$\widetilde{\Phi}(\mathbf{r},\omega) = \int_V \widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r}',\omega) d^3r' +$$

$$\oint_S \left(\widetilde{\Phi}(\mathbf{r}',\omega)\nabla\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\widetilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}\, d^2r'$$