PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 34:

Wave equation for sound

1. **Non-linear effects in traveling sound wave**
2. **Shock wave**

22	Wed, 10/17/2012	Chap. 7, 5	Moment of inertia	
	Fri, 10/19/2012		Fall break	
23	Mon, 10/22/2012	Chap. 5	Rigid body rotation	#16
24	Wed, 10/24/2012	Chap. 5	Rigid body rotation	#17
25	Fri, 10/26/2012	Chap. 5	Rigid body rotation	#18
26	Mon, 10/29/2012	Chap. 8	Waves in elastic membranes	#19
27	Wed, 10/31/2012	Chap. 9	Introduction to hydrodynamics	
28	Fri, 11/01/2012	Chap. 9	Introduction to hydrodynamics	
29	Mon, 11/05/2012	Chap. 9	Introduction to hydrodynamics	#20
30	Wed, 11/07/2012	Chap. 9	Sound waves	
31	Fri, 11/09/2012	Chap. 9	Linear sound waves	#21
32	Mon, 11/12/2012	Chap. 9	Green's function for linear sound	
33	Wed, 11/14/2012	Chap. 9	Non-linear sound	
34	Fri, 11/16/2012	Chap. 9	Non-linear sound	Take Home Exam
35	Mon, 11/19/2012	Chap. 10	Surface waves	Take Home Exam
	Wed, 11/21/2012		*Thanksgiving Holiday*	
	Fri, 11/23/2012		*Thanksgiving Holiday*	
36	Mon, 11/26/2012	Chap. 10	Surface waves	Exam due
37	Wed, 11/28/2012	Chap. 10	Surface waves	
38	Fri, 11/30/2012	Chap. 10	Surface waves	
39	Mon, 12/03/2012		Student presentations I	
40	Wed, 12/05/2012		Student presentations II	

Effects of nonlinearities in fluid equations

-- one dimensional case

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

Assume that spatial variation confined to x direction ;

assume that $\mathbf{v} = v\hat{\mathbf{x}}$ and $\mathbf{f}_{applied} = 0.$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

Expressing p in terms of ρ: $p = p(\rho)$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} \equiv c^2(\rho)\frac{\partial \rho}{\partial x} \qquad \text{where} \quad \frac{\partial p}{\partial \rho} \equiv c^2(\rho)$$

For adiabatic ideal gas :

$$c^2(\rho) = \frac{\gamma p}{\rho} = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \qquad \text{where} \quad c_0^2 \equiv \frac{\gamma p_0}{\rho_0}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

Expressing variation v in terms of variation of ρ :

$$\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial t} + v\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x} = 0$$

Some more algebra :

From Euler equation : $$\frac{\partial v}{\partial \rho}\left(\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$

From continuity equation : $$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}=-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}$$

Combined equation : $$\frac{\partial v}{\partial \rho}\left(-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$

$$\Rightarrow\left(\frac{\partial v}{\partial \rho}\right)^2=\frac{c^2(\rho)}{\rho^2} \qquad \frac{\partial v}{\partial \rho}=\pm\frac{c}{\rho}$$

$$\Rightarrow\frac{\partial \rho}{\partial t}+(v\pm c)\frac{\partial \rho}{\partial x}=0$$

Assuming adiabatic process : $c^2 = c_0^2 \left(\frac{\rho}{\rho_0} \right)^{\gamma - 1}$ $\quad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$\frac{\partial v}{\partial \rho} = \pm \frac{c}{\rho} \qquad \Rightarrow v = \pm c_0 \int_{\rho_0}^{\rho} \left(\frac{\rho'}{\rho_0} \right)^{(\gamma-1)/2} \frac{d\rho'}{\rho'}$$

$$\Rightarrow v = \pm \frac{2c_0}{\gamma - 1} \left(\left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2} - 1 \right)$$

$$\Rightarrow c = c_0 \left(\frac{\rho}{\rho_0} \right)^{(\gamma-1)/2}$$

Traveling wave solution:

Assume: $\rho = \rho_0 + f(x - u(\rho)t)$

Need to find self-consistent equations for propagation velocity $u(\rho)$ using equations

From previous derivations: $\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$

Apparently: $u(\rho) \Leftrightarrow v \pm c$

For adiabatic ideal gas and + signs:

$$u = v + c = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Visualization for particular waveform: $\rho = \rho_0 + f(x - u(\rho)t)$

Assume: $f(w) \equiv f_0 s(w)$

$$\frac{\rho}{\rho_0} = 1 + \frac{f_0}{\rho_0} s(x - ut)$$

For adiabatic ideal gas:

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1 + \frac{f_0}{\rho_0} s(x - ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Visualization continued:

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+\frac{f_0}{\rho_0}s(x-ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Plot $s(x-ut)$ for fixed t, as a function of x:

Let $w = x - ut$

$$x = w + ut = w + u(w)t$$

$$u(w) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+\frac{f_0}{\rho_0}s(w)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$