PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

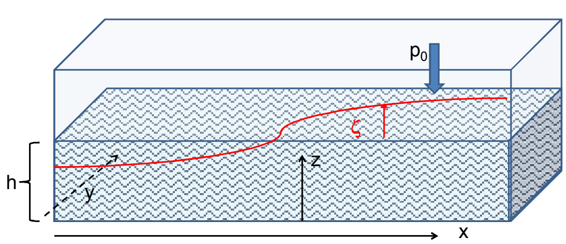
Plan for Lecture 36:

Chapter 10 in F & W: Surface waves

1. **Water waves in a shallow or deep channel – linear equations**
2. **Nonlinear effects – soliton solutions**

22	Wed, 10/17/2012	Chap. 7, 5	Moment of inertia	
	Fri, 10/19/2012		Fall break	
23	Mon, 10/22/2012	Chap. 5	Rigid body rotation	#16
24	Wed, 10/24/2012	Chap. 5	Rigid body rotation	#17
25	Fri, 10/26/2012	Chap. 5	Rigid body rotation	#18
26	Mon, 10/29/2012	Chap. 8	Waves in elastic membranes	#19
27	Wed, 10/31/2012	Chap. 9	Introduction to hydrodynamics	
28	Fri, 11/01/2012	Chap. 9	Introduction to hydrodynamics	
29	Mon, 11/05/2012	Chap. 9	Introduction to hydrodynamics	#20
30	Wed, 11/07/2012	Chap. 9	Sound waves	
31	Fri, 11/09/2012	Chap. 9	Linear sound waves	#21
32	Mon, 11/12/2012	Chap. 9	Green's function for linear sound	
33	Wed, 11/14/2012	Chap. 9	Non-linear sound	
34	Fri, 11/16/2012	Chap. 9	Non-linear sound	Take Home Exam
35	Mon, 11/19/2012	Chap. 10	Surface waves	Take Home Exam
	Wed, 11/21/2012		*Thanksgiving Holiday*	
	Fri, 11/23/2012		*Thanksgiving Holiday*	
36	Mon, 11/26/2012	Chap. 10	Surface waves	Exam due
37	Wed, 11/28/2012	Chap. 10	Surface waves	
38	Fri, 11/30/2012	Chap. 10	Surface waves	
39	Mon, 12/03/2012		Student presentations I	
40	Wed, 12/05/2012		Student presentations II	

Consider a container of water with average height h and surface h+ζ(x,y,t) (slightly different notation than last time):



Euler's equation for incompressible fluid :

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) + \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = -\nabla U - \frac{\nabla p}{\rho}$$

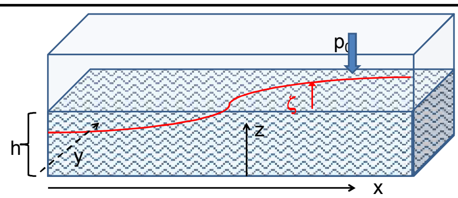
Assume that $\nabla\times\mathbf{v} = 0$ (irrotational flow) $\Rightarrow \mathbf{v} = -\nabla\Phi$

$$\Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + U + \frac{p}{\rho}\right) = 0$$

$$\Rightarrow -\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + U + \frac{p}{\rho} = \text{constant (within the fluid)}$$

For the same system, the continuity condition becomes

$$\nabla\cdot\mathbf{v} = -\nabla^2\Phi = 0$$



Within fluid : $0 \le z \le h+\zeta$

$$-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + g(z-h) = \text{constant} \quad \text{(We have absorbed}$$

$$-\nabla^2\Phi = 0 \qquad p_0 \text{ in our constant.)}$$

At surface : $z = h+\zeta$ with $\zeta = \zeta(x,y,t)$

$$\frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} + v_x\frac{\partial\zeta}{\partial x} + v_y\frac{\partial\zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}(x,y,h+\zeta,t)$$

Full equations:

Within fluid : $0 \le z \le h+\zeta$

$$-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + g(z-h) = \text{constant} \quad \text{(We have absorbed}$$

$$-\nabla^2\Phi = 0 \qquad p_0 \text{ in our constant.)}$$

At surface : $z = h+\zeta$ with $\zeta = \zeta(x,y,t)$

$$\frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} + v_x\frac{\partial\zeta}{\partial x} + v_y\frac{\partial\zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}(x,y,h+\zeta,t)$$

Linearized equations:

For $0 \le z \le h+\zeta$: $-\frac{\partial\Phi}{\partial t} + g(z-h) = 0 \qquad -\nabla^2\Phi = 0$

At surface : $z = h+\zeta \qquad \frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} = v_z(x,y,h+\zeta,t)$

$$-\frac{\partial\Phi(x,y,h+\zeta,t)}{\partial t} + g\zeta = 0$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x*:

$$\text{For} \quad 0 \le z \le h+\zeta: \quad \nabla^2\Phi = \left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}\right)\Phi(x,z,t)=0$$

$$\text{Consider and periodic waveform}: \ \Phi(x,z,t)=Z(z)\cos\left(k(x-ct)\right)$$

$$\Rightarrow \left(\frac{d^2}{dt^2}-k^2\right)Z(z)=0$$

$$\text{Boundary condition at bottom of tank}: \ v_z(x,0)=0$$

$$\Rightarrow \frac{dZ}{dz}(0)=0 \qquad Z(z)=A\cosh(kz)$$

11/26/2012 PHY 711 Fall 2012 -- Lecture 36 7

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$\text{At surface}: \ z=h+\zeta \qquad \frac{\partial\zeta}{\partial t}=v_z(x,h+\zeta,t)=-\frac{\partial\Phi(x,h+\zeta,t)}{\partial z}$$

$$-\frac{\partial\Phi(x,h+\zeta,t)}{\partial t}+g\zeta=0$$

$$-\frac{\partial^2\Phi(x,h+\zeta,t)}{\partial t^2}+g\frac{\partial\zeta}{\partial t}=-\frac{\partial^2\Phi(x,h+\zeta,t)}{\partial t^2}-g\frac{\partial\Phi(x,h+\zeta,t)}{\partial z}=0$$

$$\text{For} \quad \Phi(x,(h+\zeta),t)=A\cosh(k(h+\zeta))\cos(k(x-ct))$$

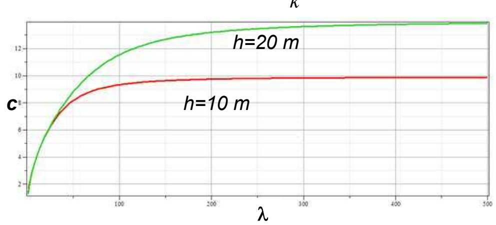
$$A\cosh(k(h+\zeta))\cos(k(x-ct))\left(k^2c^2-gk\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}\right)=0$$

11/26/2012 PHY 711 Fall 2012 -- Lecture 36 8

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2=\frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}$$

$$\text{Assuming} \ \zeta << h: \qquad c^2=\frac{g}{k}\tanh(kh)$$



11/26/2012 PHY 711 Fall 2012 -- Lecture 36 9

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 \approx \frac{g}{k}\tanh(kh) \qquad \text{For } \lambda >> h,\ c^2 \approx gh$$

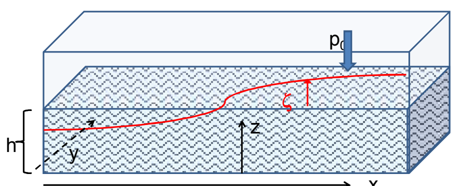
$$\Phi(x,z,t) = A\cosh(kz)\cos(k(x-ct))$$

$$\zeta(x,t) = \frac{1}{g}\frac{\partial\Phi(x,h+\zeta,t)}{\partial t} \approx \frac{kc}{g}A\cosh(kh)\sin(k(x-ct))$$

Note that for $\lambda >> h$, $c^2 \approx gh$

(solutions are consistent with previous analysis)

General problem:



Within fluid : $0 \le z \le h + \zeta$

$$-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + g(z-h) = \text{constant}$$

$$-\nabla^2\Phi = 0$$

(We have absorbed p_0 in our constant.)

At surface : $z = h + \zeta$ with $\zeta = \zeta(x,y,t)$

$$\frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} + v_x\frac{\partial\zeta}{\partial x} + v_y\frac{\partial\zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}(x,y,h+\zeta,t)$$

Some links:
- Details of soliton equations
- Maple animation
- Website – http://www.ma.hw.ac.uk/solitons/