October 9, 2012

PHY 711 – Notes on Sturm-Liouville equations –
(following Fetter and Walecka, Chap. 7)

These notes summarize some results concerning a broad class of problems related to the Sturm-Liouville system which has the general form:

$$(\mathcal{S}(x) - \lambda\sigma(x))\,\phi_0(x) \equiv \left[-\frac{d}{dx}\left(\tau(x)\frac{d}{dx}\right) + v(x) - \lambda\sigma(x)\right]\phi_0(x) = 0. \tag{1}$$

Here, $\phi_0(x)$ is an unknown function of the dependent variable x and $\tau(x)$, $v(x)$, and $\sigma(x)$ are given functions. λ is a scalar constant which can be given. In several physics examples, x respresents a one-dimensional spatial variable, but other examples require the dependent variable to represent time. We will assume that the functions are of interest in a finite range

$$a \leq x \leq b. \tag{2}$$

The full Sturm-Liouville system also involves a specification of boundary conditions. Since the equation is second order, there are in general two independent conditions which can be specified. The most common boundary conditions are listed in Eqs. 40.11-40.15 in **Fetter and Walecka** involving the end points a and b. In addition, it is sometimes the case that two independent conditions can be placed at a single end point (initial or final conditions).

Eigenfunctions and eigenvalues

For special values of λ, enumerated with an index n, we can define eigenvalues λ_n and their corresponding eigenfunctions $f_n(x)$ of the Sturm-Liouville system:

$$(\mathcal{S}(x) - \lambda_n\sigma(x))\,f_n(x) = 0. \tag{3}$$

In general, it is convenient to order the eigenvalues λ_n in order of increasing value, with λ_0 denoting the minimum eigenvalue.
We can prove as a general property of the Sturm-Liouville system, these functions are orthogonal

$$\int_a^b \sigma(x)f_n(x)f_m(x)dx = \delta_{nm}N_n, \tag{4}$$

where

$$N_n \equiv \int_a^b \sigma(x)(f_n(x))^2 dx. \tag{5}$$

It can be shown that for any reasonable function $h(x)$, defined within the interval $a \leq x \leq b$, we can expand that function as a linear combination of the eigenfunctions $\{f_n(x)\}$:

$$h(x) \approx \sum_n C_n f_n(x), \tag{6}$$

where

$$C_n = \frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx'. \tag{7}$$

These ideas lead to the notion that the set of eigenfunctions $\{f_n(x)\}$ form a "complete" set in the sense of "spanning" the space of all functions in the interval $a \le x \le b$, as summarized by the statement:

$$\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x'). \tag{8}$$

In general, there are several techniques to determine the eigenvalues $\{\lambda_n\}$ and eigenfunctions $\{f_n(x)\}$. When it is not possible to find the "exact" functions, there are several powerful approximation techniques. For example, the lowest eigenvalue can be approximated by minimizing the function

$$\lambda_0 \le \frac{\langle \tilde{h} | \mathcal{S} | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle}, \tag{9}$$

where $\tilde{h}(x)$ is a variable function which satisfies the correct boundary values. The "proof" of this inequality is based on the notion that $\tilde{h}(x)$ can in princple be expanded in terms of the (unknown) exact eigenfunctions $\{f_n(x)\}$:

$$\tilde{h}(x) = \sum_n C_n f_n(x), \tag{10}$$

where the coefficients $\{C_n\}$ can be assumed to be real. From the eigenfunction equation, we know that

$$\mathcal{S}(x) \tilde{h}(x) = \mathcal{S}(x) \sum_n C_n f_n(x) = \sum_n C_n \lambda_n \sigma(x) f_n(x). \tag{11}$$

Multiplying this equation by $\tilde{h}(x)$ and integrating over x, and using the orthogonalization properties of the eigenfunctions, we obtain

$$\langle \tilde{h} | \mathcal{S} | \tilde{h} \rangle = \int_a^b \tilde{h}(x) \mathcal{S}(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n \lambda_n. \tag{12}$$

Since

$$\langle \tilde{h} | \sigma | \tilde{h} \rangle = \int_a^b \tilde{h}(x) \sigma(x) \tilde{h}(x) dx = \sum_n |C_n|^2 N_n, \tag{13}$$

and the eigenvalue inequality becomes

$$\lambda_0 \le \frac{\sum_n |C_n|^2 N_n \lambda_n}{\sum_n |C_n|^2 N_n}. \tag{14}$$

Apparently, if all $C_n = 0$ for $n \ge 1$, then the equality is satisfied. Otherwise, the right-hand-side of the equation is larger than the left, as is consistent with the original statement. Thus Eq. (9) can be used to estimate the lowest eigenvalue λ_0 and the corresponding eigenfunction $\tilde{h}(x) \approx f_0(x)$. Higher eigenvalues can be estimated in a similar way, by also imposed orthogonalization constraints.

Green's functions

A related equation involves a Green's function $G_\lambda(x, x')$:

$$\left(\mathcal{S}(x) - \lambda \sigma(x)\right) G_\lambda(x, x') = \delta(x - x'). \tag{15}$$

This Green's function can be used to solve *inhomogeneous* Sturm-Liouville problems of the form

$$(\mathcal{S}(x) - \lambda\sigma(x))\,\phi(x) = F(x), \tag{16}$$

where now $\phi(x)$ is the function to be determined and the "forcing function" $F(x)$ is assumed to be given. In this case, if the Green's function is known, the solution can be evaluated according to
In terms of the eigenfunctions, the Green's function can be expressed as

$$G_\lambda(x, x') = \sum_n \frac{f_n(x) f_n(x')/N_n}{\lambda_n - \lambda}. \tag{17}$$

While, in general, this expansion converges, in come cases a large number of terms may be needed for accurate evaluations. With this Green's function, the solution to the inhomogeneous Eq. (16) takes the form

$$\phi(x) = \phi_0(x) + \int_a^b G(x, x') F(x') dx', \tag{18}$$

where $\phi_0(x)$ is a solution to the homogeneous equation

$$(\mathcal{S}(x) - \lambda\sigma(x))\,\phi_0(x) = 0. \tag{19}$$

Another representation of the Green's function for a Sturm-Liouville problem can be found in terms of the two independent solutions $g_a(x)$ and $g_b(x)$ of the homogeneous Eq. (19) at a given λ. For simplicity, we will assume that our full solution $\phi(x)$ satisfies the simple boundary conditions $\phi(a) = 0$ and $\phi(b) = 0$.

$$(\mathcal{S}(x) - \lambda\sigma(x))\,g_i(x) = 0, \tag{20}$$

where $i \equiv a$ for $g_a(a) = 0$ and $i \equiv b$ for $g_b(b) = 0$. We can show that

$$G_\lambda(x, x') = \frac{1}{W} g_a(x_<) g_b(x_>), \tag{21}$$

where $x_<$ means the smaller of x and x' and $x_>$ means the larger of x and x' where W represents the Wronskian:

$$W \equiv \tau(x)\left(g_b(x)\frac{dg_a(x)}{dx} - \frac{dg_b(x)}{dx} g_a(x)\right). \tag{22}$$

This form of the Green's function implies that the solution can be determined by evaluating the two integrals

$$\phi(x) = \phi_0(x) + \frac{g_b(x)}{W}\int_a^x g_a(x') F(x') dx' + \frac{g_a(x)}{W}\int_x^b g_b(x') F(x') dx'. \tag{23}$$

Laplace transforms

Laplace transforms can be used to solve initial value problems. The Laplace transform of a function $\phi(x)$ is defined as

$$\mathcal{L}_\phi(p) \equiv \int_0^\infty \mathrm{e}^{-px}\phi(x) dx. \tag{24}$$

Assuming that $\phi(x)$ is well-behaved in the interval $0 \leq x \leq \infty$, the following properties are useful:

$$\mathcal{L}_{d\phi/dx}(p) = -\phi(0) + p\mathcal{L}_\phi(p), \tag{25}$$

and

$$\mathcal{L}_{d^2\phi/dx^2}(p) = -\frac{d\phi(0)}{dx} - p\phi(0) + p^2\mathcal{L}_\phi(p). \tag{26}$$

These identities allow us to turn a differential equation for $\phi(x)$ into an algebraic equation for $\mathcal{L}_\phi(p)$. We then need to perform an inverse Laplace transform to find $\phi(x)$.
For illustration, we will consider a simple example with $\tau(x) = 1$, $\sigma(x) = 1$, $\lambda = 0$. The differential equation then becomes

$$-\frac{d^2\phi(x)}{dx^2} = F(x), \tag{27}$$

where we will take the initial conditions to be $\phi(0) = 0$ and $d\phi(0)/dx = 0$. For our example, we will also take $F(x) = F_0 \mathrm{e}^{-\gamma x}$. Multiplying, both sides of the equation by e^{-px} and integrating $0 \leq x \leq \infty$, we find

$$\mathcal{L}_\phi(p) = -\frac{F_0}{p^2(\gamma + p)}. \tag{28}$$

In general the inverse Laplace transform involves performing a contour integral, but we can use the following simple relations

$$\mathcal{L}_1 = \int_0^\infty \mathrm{e}^{-px} dx = \frac{1}{p}. \tag{29}$$

$$\mathcal{L}_x = \int_0^\infty x\mathrm{e}^{-px} dx = \frac{1}{p^2}. \tag{30}$$

$$\mathcal{L}_{\mathrm{e}^{-\gamma x}} = \int_0^\infty \mathrm{e}^{-\gamma x}\mathrm{e}^{-px} dx = \frac{1}{p + \gamma}. \tag{31}$$

Noting that

$$-\frac{F_0}{p^2(\gamma + p)} = -\frac{F_0}{\gamma^2}\left(\frac{1}{\gamma + p} - \frac{1}{p} + \frac{\gamma}{p^2}\right), \tag{32}$$

we see that the inverse Laplace transform gives us

$$\phi(x) = \frac{F_0}{\gamma^2}\left(1 - \mathrm{e}^{-\gamma x} - \gamma x\right). \tag{33}$$