PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 13:

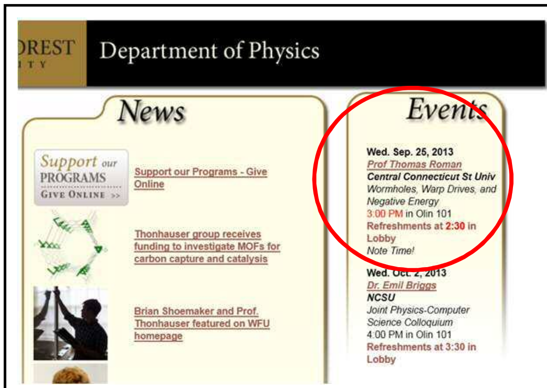
Finish reading Chapter 6

1. **Canonical transformations**
2. **Hamilton-Jacobi formalism**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/28/2013	Chap. 1	Review of basic principles;Scattering theory	#1
2	Fri, 8/30/2013	Chap. 1	Scattering theory continued	#2
3	Mon, 9/02/2013	Chap. 1	Scattering theory continued	#3
4	Wed, 9/04/2013	Chap. 2	Accelerated Coordinate Systems	#4
5	Fri, 9/06/2013	Chap. 3	Calculus of variations	#5
6	Mon, 9/09/2013	Chap. 3	Calculus of variations -- continued	
7	Wed, 9/11/2013	Chap. 3	Calculus of variations applied to Lagrangians	#6
8	Fri, 9/13/2013	Chap. 3	Lagrangian mechanics	#7
9	Mon, 9/16/2013	Chap. 3 & 6	Lagrangian mechanics	#8
10	Wed, 9/18/2013	Chap. 3 & 6	Lagrangian mechanics	#9
11	Fri, 9/20/2013	Chap. 3 & 6	Lagrangian & Hamiltonian mechanics	#10
12	Mon, 9/23/2013	Chap. 3 & 6	Hamiltonian formalism	#11
13	Wed, 9/25/2013	Chap. 3 & 6	Hamiltonian formalism	#12
14	Fri, 9/27/2013	Chap. 3 & 6	Hamiltonian formalism	#13



WFU Physics Colloquium

TITLE: Wormholes, Warp Drives, and Negative Energy

SPEAKER: Professor Thomas Roman,

Department of Mathematical Sciences ,
Central Connecticut State University

TIME: Wednesday September 25, 2013 at 3:00 **

PLACE: Room 101 Olin Physical Laboratory

** Note the early starting time

Refreshments will be served at 2:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

ABSTRACT

This talk discusses "wormholes", shortcuts through space and time, and the type of faster-than-light travel known as "warp drive". Both of these topics have been staples of science fiction for many years. However, serious scientific work has actually been done on these subjects, primarily in the 1990's, notably by such people as Kip Thorne and his students at Caltech. Such objects require a very unusual form of energy, known as "negative energy". In classical physics, all observed forms of matter have positive energy. Remarkably, the laws of quantum theory not only allow, but actually predict, the existence of negative energy. However, work done over the last several decades has shown that these same laws also limit its macroscopic effects, and therefore have profound implications as to whether wormholes and warp drives are likely to be physically realizable.

Review:
Liouville's theorem:
Imagine a collection of particles obeying the Canonical equations of motion in phase space.

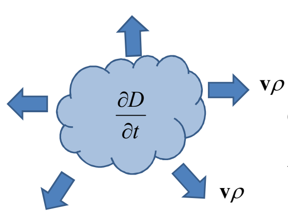
Let D denote the "distribution" of particles in phase space:

$$D = D(\{q_1 \cdots q_{3N}\}, \{p_1 \cdots p_{3N}\}, t)$$

Liouville's theorm shows that:

$$\frac{dD}{dt} = 0 \qquad \Rightarrow D \text{ is constant in time}$$

Proof of Liouville'e theorem:



Continuity equation:

$$\frac{\partial D}{\partial t} = -\nabla \cdot (\mathbf{v} D)$$

Note: in this case, the velocity is the $6N$ dimensional vector:

$$\mathbf{v} = (\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, \ldots \dot{\mathbf{r}}_N, \dot{\mathbf{p}}_1, \dot{\mathbf{p}}_2, \ldots \dot{\mathbf{p}}_N)$$

We also have a $6N$ dimensional gradient:

$$\nabla = (\nabla_{\mathbf{r}_1}, \nabla_{\mathbf{r}_2}, \ldots \nabla_{\mathbf{r}_N}, \nabla_{\mathbf{p}_1}, \nabla_{\mathbf{p}_2}, \ldots \nabla_{\mathbf{p}_N})$$

$$\frac{\partial D}{\partial t} = -\nabla \cdot (\mathbf{v}D)$$

$$= -\sum_{j=1}^{3N}\left[\frac{\partial}{\partial q_j}(\dot{q}_j D) + \frac{\partial}{\partial p_j}(\dot{p}_j D)\right]$$

$$= -\sum_{j=1}^{3N}\left[\frac{\partial D}{\partial q_j}\dot{q}_j + \frac{\partial D}{\partial p_j}\dot{p}_j\right] - D\sum_{j=1}^{3N}\left[\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j}\right]$$

$$\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j} = \frac{\partial^2 H}{\partial q_j \partial p_j} + \left(-\frac{\partial^2 H}{\partial p_j \partial q_j}\right) = 0$$

0

$$\frac{\partial D}{\partial t} = -\sum_{j=1}^{3N}\left[\frac{\partial D}{\partial q_j}\dot{q}_j + \frac{\partial D}{\partial p_j}\dot{p}_j\right] - D\sum_{j=1}^{3N}\left[\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \dot{p}_j}{\partial p_j}\right]$$

$$\frac{\partial D}{\partial t} = -\sum_{j=1}^{3N}\left[\frac{\partial D}{\partial q_j}\dot{q}_j + \frac{\partial D}{\partial p_j}\dot{p}_j\right]$$

$$\Rightarrow \frac{\partial D}{\partial t} + \sum_{j=1}^{3N}\left[\frac{\partial D}{\partial q_j}\dot{q}_j + \frac{\partial D}{\partial p_j}\dot{p}_j\right] = \frac{dD}{dt} = 0$$

Notion of "Canonical" distributions

$$q_\sigma = q_\sigma(\{Q_1 \cdots Q_n\}, \{P_1 \cdots P_n\}, t) \qquad \text{for each } \sigma$$

$$p_\sigma = p_\sigma(\{Q_1 \cdots Q_n\}, \{P_1 \cdots P_n\}, t) \qquad \text{for each } \sigma$$

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\}, \{p_\sigma\}, t) =$$

$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \widetilde{H}(\{Q_\sigma\}, \{P_\sigma\}, t) + \frac{d}{dt}F(\{q_\sigma\}, \{Q_\sigma\}, t)$$

Apply Hamilton's principle :

$$\delta \int_{t_i}^{t_f}\left[\sum_\sigma P_\sigma \dot{Q}_\sigma - \widetilde{H}(\{Q_\sigma\}, \{P_\sigma\}, t) + \frac{d}{dt}F(\{q_\sigma\}, \{Q_\sigma\}, t)\right]dt = 0$$

$$\dot{Q}_\sigma = \frac{\partial \widetilde{H}}{\partial P_\sigma} \qquad \dot{P}_\sigma = -\frac{\partial \widetilde{H}}{\partial Q_\sigma}$$

Note that it is conceivable that if we were extraordinarily clever, we could find all of the constants of the motion!

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t)$$

$$\frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t) = \sum_\sigma \left(\left(\frac{\partial F}{\partial q_\sigma}\right)\dot{q}_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\dot{Q}_\sigma \right) + \frac{\partial F}{\partial t}$$

$$\sum_\sigma \left(p_\sigma - \left(\frac{\partial F}{\partial q_\sigma}\right)\right)\dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma \left(P_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\right)\dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{\partial F}{\partial t}$$

9/25/2013 PHY 711 Fall 2013 -- Lecture 13 10

$$\sum_\sigma \left(p_\sigma - \left(\frac{\partial F}{\partial q_\sigma}\right)\right)\dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma \left(P_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\right)\dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{\partial F}{\partial t}$$

$$\Rightarrow p_\sigma = \left(\frac{\partial F}{\partial q_\sigma}\right) \qquad P_\sigma = -\left(\frac{\partial F}{\partial Q_\sigma}\right)$$

$$\Rightarrow \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) = H(\{q_\sigma\},\{p_\sigma\},t) + \frac{\partial F}{\partial t}$$

9/25/2013 PHY 711 Fall 2013 -- Lecture 13 11

Note that it is conceivable that if we were extraordinarily clever, we could find all of the constants of the motion!

$$\dot{Q}_\sigma = \frac{\partial \tilde{H}}{\partial P_\sigma} \qquad \dot{P}_\sigma = -\frac{\partial \tilde{H}}{\partial Q_\sigma}$$

$$\text{Suppose : } \dot{Q}_\sigma = \frac{\partial \tilde{H}}{\partial P_\sigma} = 0 \text{ and } \dot{P}_\sigma = -\frac{\partial \tilde{H}}{\partial Q_\sigma} = 0$$

$$\Rightarrow Q_\sigma, P_\sigma \quad \text{are constants of the motion}$$

Possible solution – Hamilton-Jacobi theory:

$$\text{Suppose : } F(\{q_\sigma\},\{Q_\sigma\},t) \Rightarrow -\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)$$

9/25/2013 PHY 711 Fall 2013 -- Lecture 13 12

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$
$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \widetilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}\left(-\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$
$$= -\widetilde{H}(\{Q_\sigma\},\{P_\sigma\},t) - \sum_\sigma \dot{P}_\sigma Q_\sigma + \sum_\sigma\left(\frac{\partial S}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial S}{\partial P_\sigma}\dot{P}_\sigma\right) + \frac{\partial S}{\partial t}$$

Solution :

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \qquad Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$
$$\widetilde{H}(\{Q_\sigma\},\{P_\sigma\},t) = H(\{q_\sigma\},\{p_\sigma\},t) + \frac{\partial S}{\partial t}$$

When the dust clears :

Assume $\{Q_\sigma\},\{P_\sigma\},\widetilde{H}$ are constants; choose $\widetilde{H}=0$

Need to find $S(\{q_\sigma\},\{P_\sigma\},t)$

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \qquad Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$
$$\Rightarrow H\left(\{q_\sigma\},\left\{\frac{\partial S}{\partial q_\sigma}\right\},t\right) + \frac{\partial S}{\partial t} = 0$$

Note : S is the "action":

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$
$$\underbrace{\sum_\sigma P_\sigma \dot{Q}_\sigma}_{0} - \underbrace{\widetilde{H}(\{Q_\sigma\},\{P_\sigma\},t)}_{0} + \frac{d}{dt}\left(-\underbrace{\sum_\sigma P_\sigma Q_\sigma}_{0} + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$
$$\underbrace{\sum_\sigma P_\sigma \dot{Q}_\sigma}_{0} - \underbrace{\widetilde{H}(\{Q_\sigma\},\{P_\sigma\},t)}_{0} + \frac{d}{dt}\left(-\underbrace{\sum_\sigma P_\sigma Q_\sigma}_{0} + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

$$\int_{t_i}^{t_f}\left(\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t)\right)dt = \int_{t_i}^{t_f}\left(\frac{d}{dt}\left(S(\{q_\sigma\},\{P_\sigma\},t)\right)\right)dt$$
$$= S(\{q_\sigma\},\{P_\sigma\},t)\Big|_{t_i}^{t_f}$$

Differential equation for ***S***:

$$H\left(\{q_\sigma\},\left\{\frac{\partial S}{\partial q_\sigma}\right\},t\right)+\frac{\partial S}{\partial t}=0$$

$$\text{Example:}\qquad H(\{q\},\{p\},t)=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2$$

$$\text{Hamilton - Jacobi Eq:}\quad H\left(\{q\},\left\{\frac{\partial S}{\partial q}\right\},t\right)+\frac{\partial S}{\partial t}=0$$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+\frac{1}{2}m\omega^2q^2+\frac{\partial S}{\partial t}=0$$

$$\text{Assume:}\qquad S(q,t)\equiv W(q)-Et\qquad\qquad (E\quad\text{constant})$$

Continued:

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+\frac{1}{2}m\omega^2q^2+\frac{\partial S}{\partial t}=0$$

$$\text{Assume:}\qquad S(q,t)\equiv W(q)-Et\qquad\qquad (E\quad\text{constant})$$

$$\frac{1}{2m}\left(\frac{dW}{dq}\right)^2+\frac{1}{2}m\omega^2q^2=E$$

$$\frac{dW}{dq}=\sqrt{2mE-(m\omega)^2q^2}$$

$$W(q)=\int\sqrt{2mE-(m\omega)^2q^2}\,dq$$

Continued:

$$W(q)=\int\sqrt{2mE-(m\omega)^2q^2}\,dq$$

$$=\frac{1}{2}q\sqrt{2mE-(m\omega)^2q^2}+\frac{E}{\omega}\sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}}\right)+C$$

$$S(q,E,t)=\frac{1}{2}q\sqrt{2mE-(m\omega)^2q^2}+\frac{E}{\omega}\sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}}\right)-Et$$

$$\frac{\partial S}{\partial E}=Q=\frac{1}{\omega}\sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}}\right)-t$$

$$\Rightarrow q(t)=\frac{\sqrt{2mE}}{m\omega}\sin(\omega(t+Q))$$