PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 26:

Traveling and standing waves in 1 and 2 dimensions

1. **Brief review of 1-d from Chap. 7**
2. **Waves in 2-d elastic membranes – Chap. 8**

6	Mon, 9/09/2013	Chap. 3	Calculus of variations -- continued	
7	Wed, 9/11/2013	Chap. 3	Calculus of variations applied to Lagrangians	#6
8	Fri, 9/13/2013	Chap. 3	Lagrangian mechanics	#7
9	Mon, 9/16/2013	Chap. 3 & 6	Lagrangian mechanics	#8
10	Wed, 9/18/2013	Chap. 3 & 6	Lagrangian mechanics	#9
11	Fri, 9/20/2013	Chap. 3 & 6	Lagrangian & Hamiltonian mechanics	#10
12	Mon, 9/23/2013	Chap. 3 & 6	Hamiltonian formalism	#11
13	Wed, 9/25/2013	Chap. 3 & 6	Hamiltonian formalism	#12
14	Fri, 9/27/2013	Chap. 3 & 6	Hamiltonian formalism	#13
15	Mon, 9/30/2013	Chap. 4	Small Oscillations	#14
16	Wed, 10/02/2013	Chap. 4	Small Oscillations	
17	Fri, 10/04/2013	Chap. 4	Small Oscillations	#15
18	Mon, 10/07/2013	Chap. 4 & 7	Small Oscillations and waves	#16
19	Wed, 10/09/2013	Chap. 7	Wave equation	
	Fri, 10/11/2013		No class (Fall Break)	
20	Mon, 10/14/2013	Chap. 7	Wave equation (Presentation topic due)	#17
21	Wed, 10/16/2013	Chap. 7	Mathematical methods	#18
22	Fri, 10/18/2013	Chap. 7	Mathematical methods	#19
23	Mon, 10/21/2013	Chap. 5	Rigid rotations	#20
24	Wed, 10/23/2013	Chap. 5	Rigid rotations	#21
25	Fri, 10/25/2013	Chap. 5	Rigid rotations	
	Mon, 10/28/2013	No class	Take-home exam	
	Wed, 10/30/2013	No class	Take-home exam	
	Fri, 11/01/2013	No class	Take-home exam	
26	Mon, 11/04/2013	Chap. 8	Oscillations in two-dimensional membranes	Take-home exam due

Review of wave equation in one-dimension – here $\mu(x,t)$ can describe either a longitudinal or transverse wave.

Traveling wave solutions --

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

satisfies the wave equation.

Initial value problem : $\mu(x,0) = \phi(x)$ and $\frac{\partial \mu}{\partial t}(x,0) = \psi(x)$

then : $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

For each x, find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$g(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\left(\phi(x-ct) + \phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

Standing wave solutions of wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

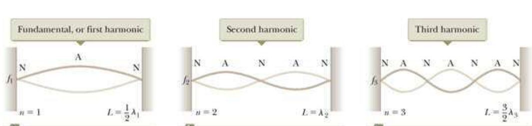
with $\mu(0,t) = \mu(L,t) = 0$.

Assume : $\mu(x,t) = \Re\left(e^{-i\omega t}\rho(x)\right)$

where $\frac{d^2\rho(x)}{dx^2} + k^2\rho(x) = 0$ $\qquad k = \frac{\omega}{c}$

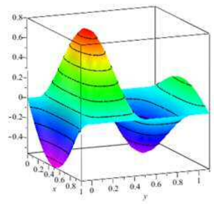
$$\rho_\nu(x) = A\sin\left(\frac{\nu\pi x}{L}\right)$$

$$k_\nu = \frac{\nu\pi}{L} \qquad \omega_\nu = ck_\nu$$



Extension of ideas to wave motion on a two-dimensional surface – elastic membrane (transverse wave; linear regime).

u(x,y,t)



Lagrangian density : $\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right)$

$$L = \int \mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) dxdy$$

Hamilton's principle :

$$\delta \int_{t_1}^{t_2} L dt = 0$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial t)} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial x)} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial y)} = 0$$

Lagrangian density for elastic membrane with constant σ and τ :

$$\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) = \frac{1}{2}\sigma\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau(\nabla u)^2$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial t)} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial x)} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial(\partial u/\partial y)} = 0$$

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Two - dimensional wave equation :

$$\frac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0 \qquad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions :

$$u(x, y, t) = \Re\left(e^{-i\omega t}\rho(x, y)\right)$$

$$\left(\nabla^2 + k^2\right)\rho(x, y) = 0 \qquad \text{where } k = \frac{\omega}{c}$$

Consider a rectangular boundary:

a

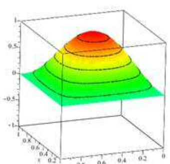
b

Clamped boundary conditions:

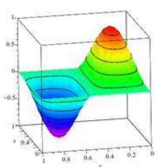
$$\rho(0,y)=\rho(a,y)=\rho(x,0)=\rho(x,b)=0$$

$$\Rightarrow \rho_{mn}(x,y)=A\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

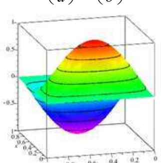
$$k_{mn}{}^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn}=ck_{mn}$$



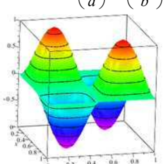
$$k_{11}{}^2=\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2$$



$$k_{12}{}^2=\left(\frac{\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2$$



$$k_{21}{}^2=\left(\frac{2\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2$$



$$k_{22}{}^2=\left(\frac{2\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2$$

More general boundary conditions:

$\tau\nabla u\big|_b=\kappa u\big|_b$ represents boundard side constrained with spring

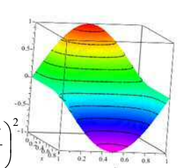
$\tau\nabla u\big|_b=0$ represents "free" side

Mixed boundary conditions:

$$\rho(x,0)=\rho(x,b)=\frac{\partial\rho(0,y)}{\partial x}=\frac{\partial\rho(a,y)}{\partial x}=0$$

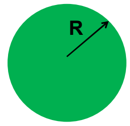
$$\Rightarrow \rho_{mn}(x,y)=A\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}{}^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2 \qquad \omega_{mn}=ck_{mn}$$



$$k_{11}{}^2=\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2$$

Consider a circular boundary:



Clamped boundary conditions for $\rho(r,\varphi)$:
$\rho(R,\varphi)=0$

$$\left(\nabla^2+k^2\right)\rho(r,\varphi)=0 \qquad \text{where } k=\frac{\omega}{c}$$

In cylindrical coordinate system

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

Assume: $\rho(r,\varphi)=f(r)\Phi(\varphi)$

Let: $\Phi(\varphi)=e^{im\varphi}$

Note: $\Phi(\varphi)=\Phi(\varphi+2\pi)$

$\Rightarrow m=$ integer

Consider circular boundary -- continued

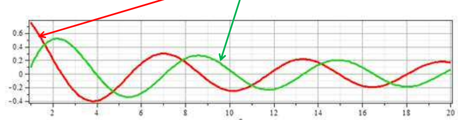
Differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$\Rightarrow$ Bessel equation of integer order with transcendential solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$



Some properties of Bessel functions

Asending series: $J_m(z)=\left(\frac{z}{2}\right)^m\sum_{j=0}^{\infty}\frac{(-1)^j}{j!(j+m)!}\left(\frac{z}{2}\right)^{2j}$

Recursion relations: $J_{m-1}(z)+J_{m+1}(z)=\frac{2m}{z}J_m(z)$

$$J_{m-1}(z)-J_{m+1}(z)=2\frac{dJ_m(z)}{dz}$$

Asymptotic form: $J_m(z)\xrightarrow[z>>1]{}\sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{m\pi}{2}-\frac{\pi}{4}\right)$

Zeros of Bessel functions $J_m(z_{mn})=0$

$m=0$: $z_{0n}=2.406,\ 5.520,\ 8.654,...$

$m=1$: $z_{1n}=3.832,\ 7.016,\ 10.173,...$

$m=2$: $z_{2n}=5.136,\ 8.417,\ 11.620,...$

Some properties of Bessel functions -- continued

Note: It is possible to prove the following identity for the functions $J_m\left(\frac{z_{mn}}{R}r\right)$:

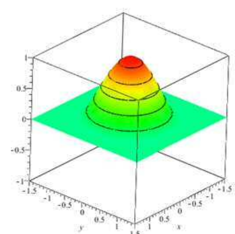
$$\int_0^R J_m\left(\frac{z_{mn}}{R}r\right)J_m\left(\frac{z_{mn'}}{R}r\right)r\,dr = \frac{R^2}{2}\left(J_{m+1}(z_{mn})\right)^2\delta_{nn'}$$

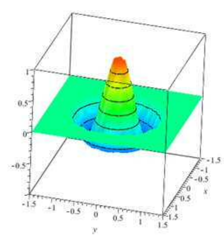
Returning to differential equation for radial function:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+k^2\right)f(r)=0$$

$$\Rightarrow\ f_{mn}(r)=AJ_m\left(\frac{z_{mn}}{R}r\right); \qquad k_{mn}=\frac{z_{mn}}{R}$$

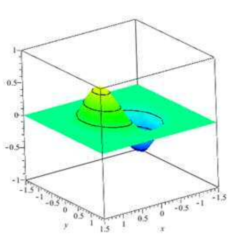
$$\rho_{01}(r,\varphi)=f_{01}(r)=AJ_0\left(\frac{z_{01}}{R}r\right) \qquad \rho_{02}(r,\varphi)=f_{02}(r)=AJ_0\left(\frac{z_{02}}{R}r\right)$$

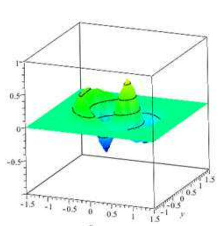




$$k_{01}=\frac{2.406}{R} \qquad k_{02}=\frac{5.520}{R}$$

$$\rho_{11}(r,\varphi)=f_{11}(r)\cos(\varphi)=AJ_1\left(\frac{z_{11}}{R}r\right)\cos(\varphi) \qquad \rho_{12}(r,\varphi)=f_{12}(r)\cos(\varphi)=AJ_1\left(\frac{z_{12}}{R}r\right)\cos(\varphi)$$





$$k_{11}=\frac{3.832}{R} \qquad k_{12}=\frac{7.016}{R}$$

Maple server evaluation project

- Request from Rick Matthews, *Associate Provost for Technology and Information Systems*
- The University is exploring the possibility of making Maple available on a server rather than on individual computers
  - Need students to exercise the software
  - Need students to assess the performanc
  - Need students to identify issues
- Entirely voluntary

Possible pizza party or ??? rewards

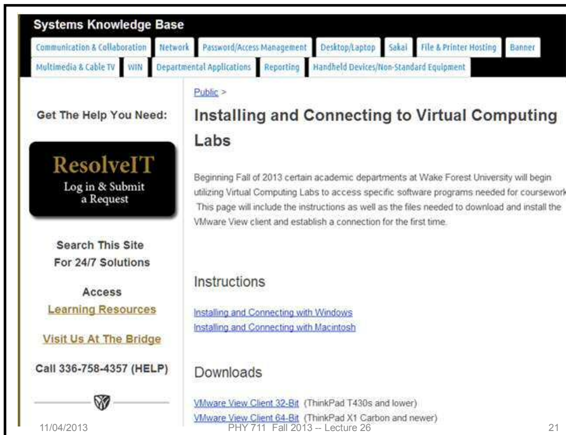
Introduction to algebraic manipulation software

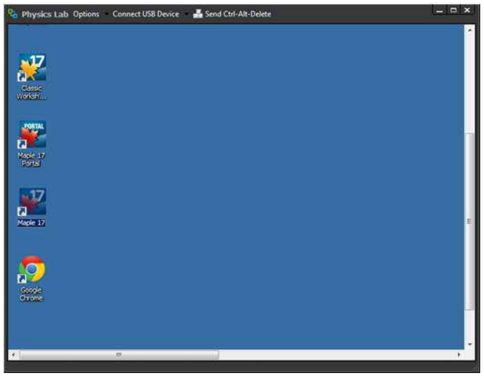
Maple 17

Available from the webpage:
http://help.wfu.edu/public/vcl

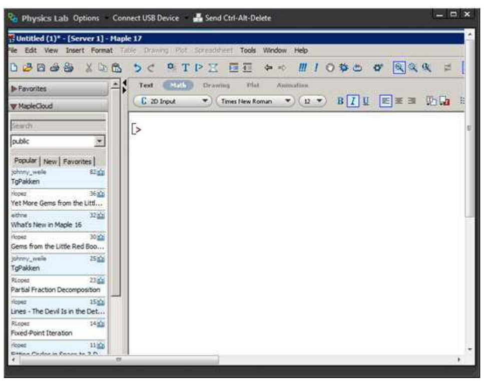
If you have any trouble with this installation and setup, please contact Ching-Wan Yip at yipcw@wfu.edu

Note: This is a remote server that the university is testing. If you are off campus you will need to use VPN to gain access.





11/04/2013 PHY 711 Fall 2013 -- Lecture 26 22



11/04/2013 PHY 711 Fall 2013 -- Lecture 26 23