PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 31:

Wave equation for sound

1. **Non-linear effects in traveling sound wave**
2. **Shock wave**

22	Fri, 10/18/2013	Chap. 7	Mathematical methods	#19
23	Mon, 10/21/2013	Chap. 5	Rigid rotations	#20
24	Wed, 10/23/2013	Chap. 5	Rigid rotations	#21
25	Fri, 10/25/2013	Chap. 5	Rigid rotations	
	Mon, 10/28/2013	No class	Take-home exam	
	Wed, 10/30/2013	No class	Take-home exam	
	Fri, 11/01/2013	No class	Take-home exam	
26	Mon, 11/04/2013	Chap. 8	Oscillations in two-dimensional membranes	Take-home exam due
27	Wed, 11/06/2013	Chap. 9	Physics of fluids	#22
28	Fri, 11/08/2013	Chap. 9	Physics of fluids	#23
29	Mon, 11/11/2013	Chap. 9	Sound Waves	#24
30	Wed, 11/13/2013	Chap. 9	Sound Waves	#25
31	Fri, 11/15/2013	Chap. 9	Non linear effects in Sound	#26
32	Mon, 11/18/2013	Chap. 10	Surface waves	
33	Wed, 11/20/2013	Chap. 10	Surface waves	
34	Fri, 11/22/2013			
35	Mon, 11/25/2013			
	Wed, 11/27/2013		*Thanksgiving Holiday*	
	Fri, 11/29/2013		*Thanksgiving Holiday*	
36	Mon, 12/02/2013		Student presentations I	
37	Wed, 12/04/2013		Student presentations II	
38	Fri, 12/06/2013		Student presentations III	
	Mon, 12/09/2013		Begin Take-home final	

Effects of nonlinearities in fluid equations
-- one dimensional case

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

Assume that spatial variation confined to x direction ;
assume that $\mathbf{v} = v\hat{\mathbf{x}}$ and $\mathbf{f}_{applied} = 0$.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

Expressing p in terms of ρ: $p = p(\rho)$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} \equiv c^2(\rho)\frac{\partial \rho}{\partial x} \qquad \text{where } \frac{\partial p}{\partial \rho} \equiv c^2(\rho)$$

For adiabatic ideal gas :

$$c^2(\rho) = \frac{\gamma p}{\rho} = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \quad \text{where } c_0^2 \equiv \frac{\gamma p_0}{\rho_0}$$

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$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

Expressing variation v in terms of $v(\rho)$:

$$\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial t} + v\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x} = 0$$

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Some more algebra :

From Euler equation : $\frac{\partial v}{\partial \rho}\left(\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x}\right) + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$

From continuity equation : $\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} = -\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}$

Combined equation : $\frac{\partial v}{\partial \rho}\left(-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}\right) + \frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x} = 0$

$$\Rightarrow \left(\frac{\partial v}{\partial \rho}\right)^2 = \frac{c^2(\rho)}{\rho^2} \qquad \frac{\partial v}{\partial \rho} = \pm\frac{c}{\rho}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

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Assuming adiabatic process : $c^2 = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ $\quad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$\frac{\partial v}{\partial \rho} = \frac{dv}{d\rho} = \pm \frac{c}{\rho} \quad \Rightarrow v = \pm c_0 \int_{\rho_0}^{\rho} \left(\frac{\rho'}{\rho_0}\right)^{(\gamma-1)/2} \frac{d\rho'}{\rho'}$$

$$\Rightarrow v = \pm \frac{2c_0}{\gamma-1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right)$$

$$\Rightarrow c = c_0 \left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2}$$

Summary :

$$\frac{dv}{d\rho} = \pm \frac{c}{\rho}$$

$$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Assuming adiabatic process : $c^2 = c_0^2 \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ $\quad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$c = c_0 \left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} \qquad v = \pm \frac{2c_0}{\gamma-1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right)$$

Traveling wave solution:

Assume : $\rho = \rho_0 + f(x - u(\rho)t)$

Need to find self - consistent equations for propagation velocity $u(\rho)$ using equations

From previous derivations : $\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$

Apparently : $u(\rho) \Leftrightarrow v \pm c$

For adiabatic ideal gas and + signs :

$$u = v + c = c_0 \left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Traveling wave solution -- continued:

$$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Assume: $\rho = \rho_0 + f(x - u(\rho)t) = \rho_0 + f(x - (v \pm c)t)$

For adiabatic ideal gas and + signs:

$$u = v + c = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Solution in linear approxiation:

$$u = v + c \approx v_0 + c_0 = c_0\left(\frac{\gamma+1}{\gamma-1} - \frac{2}{\gamma-1}\right) = c_0$$

$$\Rightarrow \rho = \rho_0 + f(x - c_0 t)$$

Traveling wave solution -- full non-linear case:

Visualization for particular waveform: $\rho = \rho_0 + f(x - u(\rho)t)$

Assume: $f(w) \equiv f_0 s(w)$

$$\frac{\rho}{\rho_0} = 1 + \frac{f_0}{\rho_0}s(x - ut)$$

For adiabatic ideal gas:

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1 + \frac{f_0}{\rho_0}s(x-ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Visualization continued:

$$u = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1 + \frac{f_0}{\rho_0}s(x-ut)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

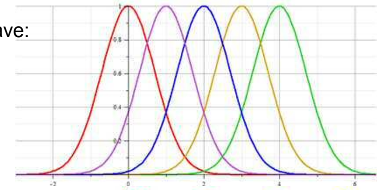
Plot $s(x - ut)$ for fixed t, as a function of x:

Let $w = x - ut$

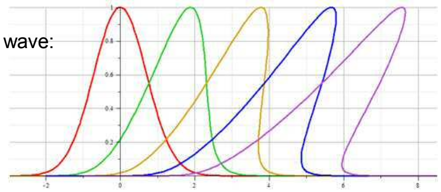
$x = w + ut = w + u(w)t$

$$u(w) = c_0\left(\frac{\gamma+1}{\gamma-1}\left(1 + \frac{f_0}{\rho_0}s(w)\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Linear wave:

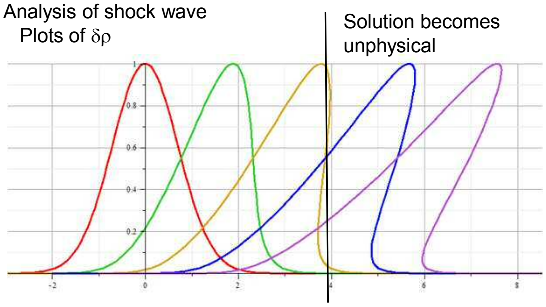


Non-linear wave:

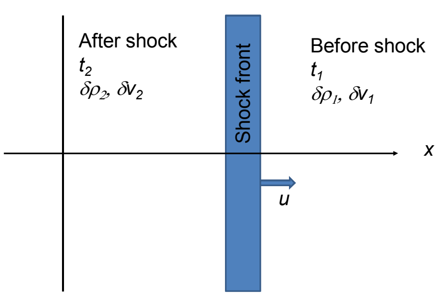


Analysis of shock wave
Plots of $\delta\rho$

Solution becomes unphysical



Analysis of shock wave -- continued



Analysis of shock wave – continued

While analysis in the shock region is complicated, we can use conservation laws to analyze regions 1 and 2

Assume $\rho(x,t) = \rho(x-ut)$

$p(x,t) = \rho(x-ut)$

$v(x,t) = \rho(x-ut)$

Continuity equation :

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 = \frac{\partial(\rho v - \rho u)}{\partial x} \quad \Rightarrow (v_2 - u)\rho_2 = (v_1 - u)\rho_1$$

Conservation of momentum :

$$\Rightarrow p_2 + \rho_2(v_2 - u)^2 = p_1 + \rho_1(v_1 - u)^2$$

Analysis of shock wave – continued

For adiabatic ideal gas, also considering energy conservation:

$$\frac{\rho_2}{\rho_1} = \frac{\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1} + 1}{\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}} \leq \frac{\gamma+1}{\gamma-1}$$

Velocity relationships:

$$\frac{(v_1-u)^2}{c_1^2} = \frac{1}{2\gamma}\left(\gamma - 1 + (\gamma+1)\frac{p_2}{p_1}\right) \quad \frac{(v_2-u)^2}{c_2^2} = \frac{1}{2\gamma}\left(\gamma - 1 + (\gamma+1)\frac{p_1}{p_2}\right)$$

Analysis of shock wave – continued

For adiabatic ideal gas, entropy considerations::

Internal energy density : $\varepsilon = \frac{p}{(\gamma-1)\rho} = C_V T$

First law of thermo : $d\varepsilon = Tds - pd\left(\frac{1}{\rho}\right)$

$$ds = \frac{1}{T}\left(d\left(\frac{p}{(\gamma-1)\rho}\right) - pd\left(\frac{1}{\rho}\right)\right) = C_V d \ln\left(\frac{p}{\rho^\gamma}\right)$$

$$s = C_V \ln\left(\frac{p}{\rho^\gamma}\right) + (\text{constant})$$

$$s_2 - s_1 = C_V \ln\left(\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^\gamma\right)$$