PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 6:

Start reading Chapter 3 –

First focusing on the "calculus of variation"

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/28/2013	Chap. 1	Review of basic principles;Scattering theory	#1
2	Fri, 8/30/2013	Chap. 1	Scattering theory continued	#2
3	Mon, 9/02/2013	Chap. 1	Scattering theory continued	#3
4	Wed, 9/04/2013	Chap. 2	Accelerated Coordinate Systems	#4
5	Fri, 9/06/2013	Chap. 3	Calculus of variations	#5
6	Mon, 9/09/2013	Chap. 3	Calculus of variations -- continued	

Comment on HW #3

PHY 711 -- Assignment #3

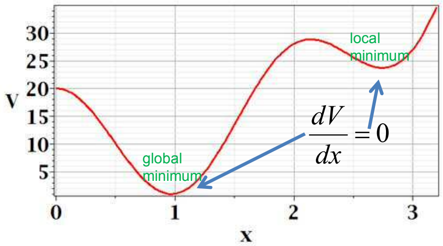
Sept. 2, 2013

Finish reading Chapter 1 in **Fetter & Walecka**.

1. In the last lecture, we derived the differential cross section for the elastic scattering of two hard spheres -- $d\sigma/d\Omega|_{CM}(\theta) = D^2/4$, where D is the sum of the radii of the two spheres. Now suppose that in lab frame of reference, the incident mass (m_1) has an initial velocity v_1 and the target mass (m_2) is at rest.
   a. Find the relationship between the center of mass scattering angle θ to the lab frame scattering angle χ.
   b. Find the relationship between the lab and CM differential cross sections.
   c. Evaluate these expressions for the case that $m_1=m_2$. Check your results to make sure that the total scattering cross section is the same in the two frames of reference.
2. Extra credit: Work Problem 1.16 in **Fetter and Walecka**

In Chapter 3, the notion of Lagrangian dynamics is developed; reformulating Newton's laws in terms of minimization of related functions. In preparation, we need to develop a mathematical tool known as "the calculus of variation".

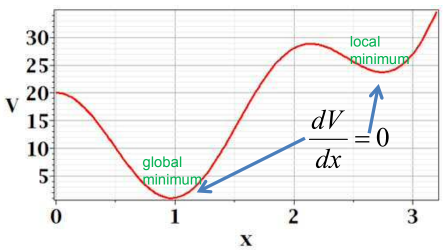
Minimization of a simple function



Minimization of a simple function

Given a function $V(x)$, find the value(s) of x for which $V(x)$ is minimized (or maximized).

Necessary condition : $\frac{dV}{dx} = 0$



Functional minimization

Consider a family of functions $y(x)$, with the end points

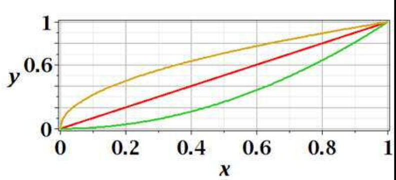
$y(x_i) = y_i$ and $y(x_f) = y_f$ and a function $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$.

Necessary condition : $\delta L = 0$

Example :

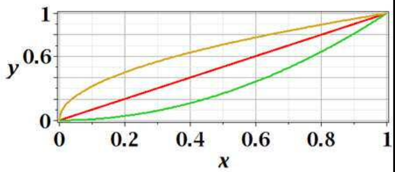
$$L = \int_{(0,0)}^{1,1} \sqrt{(dx)^2 + (dy)^2}$$



Example:

$$L = \int_{(0,0)}^{1,1} \sqrt{(dx)^2 + (dy)^2}$$

$$= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



Sample functions:

$$y_1(x) = \sqrt{x} \qquad L = \int_0^1 \sqrt{1 + \frac{1}{4x}}\, dx = 1.4789$$

$$y_2(x) = x \qquad L = \int_0^1 \sqrt{1+1}\, dx = \sqrt{2} = 1.4142$$

$$y_2(x) = x^2 \qquad L = \int_0^1 \sqrt{1 + 4x^2}\, dx = 1.4789$$

Calculus of variation example for a pure integral functions

Find the function $y(x)$ which extremizes $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right)$

where $L\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) \equiv \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx.$

Necessary condition: $\delta L = 0$

At any x, let $y(x) \rightarrow y(x) + \delta y(x)$

$$\frac{dy(x)}{dx} \rightarrow \frac{dy(x)}{dx} + \delta \frac{dy(x)}{dx}$$

Formally:

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y} \delta\left(\frac{dy}{dx}\right)\right]\right] dx.$$

After some derivations, we find

$$\delta L = \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} \delta y + \left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y} \delta\left(\frac{dy}{dx}\right)\right]\right] dx$$

$$= \int_{x_i}^{x_f} \left[\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right]\right] \delta y\, dx = 0 \quad \text{for all } x_i \le x \le x_f$$

$$\Rightarrow \left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0 \quad \text{for all } x_i \le x \le x_f$$

Example : End points -- $y(0)=0;\ y(1)=1$

$$L=\int_0^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$

Solution :

$$\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=K \qquad \frac{dy}{dx}=K'\equiv\frac{K}{\sqrt{1-K^2}}$$

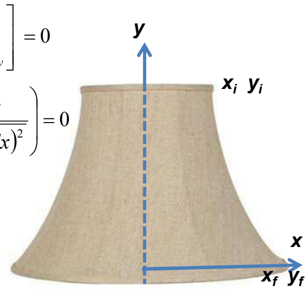
$$\Rightarrow y(x)=x$$

Example :

$$A=2\pi\int_{x_i}^{x_f}x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)=x\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$



$$-\frac{d}{dx}\left(\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}\right)=0$$

$$\frac{x\,dy/dx}{\sqrt{1+(dy/dx)^2}}=K_1$$

$$\frac{dy}{dx}=-\frac{1}{\sqrt{\left(\frac{x}{K_1}\right)^2-1}}$$

$$\Rightarrow y(x)=K_2-K_1\ln\left(x+\sqrt{x^2-K_1^2}\right)$$

Another example:
(Courtesy of F. B. Hildebrand, Methods of Applied Mathematics)

Consider all curves $y(x)$ with $y(0)=0$ and $y(1)=1$ that minimize the integral :

$$I=\int_0^1\left(\left(\frac{dy}{dx}\right)^2-ay^2\right)dx \quad \text{for constant } a>0$$

Euler - Lagrange equation :

$$\frac{d^2y}{dx^2}+ay=0$$

$$\Rightarrow y(x)=\frac{\sin\left(\sqrt{a}x\right)}{\sin\left(\sqrt{a}\right)}$$

Review : for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}-\frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]=0$$

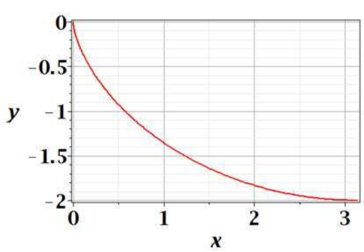
Note that for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

$$\frac{df}{dx}=\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx}+\left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx}+\left(\frac{\partial f}{\partial x}\right)$$

$$=\left(\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)\right)\frac{dy}{dx}+\left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx}+\left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f-\frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right)=\left(\frac{\partial f}{\partial x}\right)$$

Brachistochrone problem: (solved by Newton in 1696)
http://mathworld.wolfram.com/BrachistochroneProblem.html



A particle of weight *mg* travels frictionlessly down a path of shape *y(x)*. What is the shape of the path *y(x)* that minimizes the travel time from *y(0)=0* to *y(π)=-2*?

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{2gy}}\, dx \quad \text{because } \tfrac{1}{2}mv^2 = mgy$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = 0$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

Note that for the original form of Euler - Lagrange equation :

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0,$$

differential equation is more complicated :

$$-\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{y^3}} - \frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$