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# PHY 711 – Notes on Hydrodynamics – ("Solitary Waves"[1])

## Basic assumptions

We assume that we have in incompressible fluid ($\rho$ = constant) a velocity potential of the form $\Phi(x, z, t)$, where

$$\mathbf{v}(x, z, t) = -\nabla\Phi(x, z, t). \tag{1}$$

The surface of the fluid is described by $z = h + \zeta(x, t)$. It is assumed that the fluid is contained in a structure (lake, river, swimming pool, etc.) with a structureless bottom defined by the $z = 0$ plane and filled to an equilibrium height of $z = h$. The functions $\Phi(x, z, t)$ and $\zeta(x, t)$ satisfy the following conditions.

The continuity equation ($\nabla \cdot \mathbf{v} = 0$) becomes the Laplace equation for $\Phi(x, z, t)$:

$$\frac{\partial^2\Phi(x, z, t)}{\partial x^2} + \frac{\partial^2\Phi(x, z, t)}{\partial z^2} = 0 \tag{2}$$

If we assume irrotational flow, $\nabla \times \mathbf{v} = 0$, we also have the Bernoulli equation which at the top surface ($z = h + \zeta(x, t)$) takes the form:

$$-\frac{\partial\Phi(x, z, t)}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\Phi(x, z, t)}{\partial x}\right)^2 + \left(\frac{\partial\Phi(x, z, t)}{\partial z}\right)^2\right] + g(z - h) = 0. \text{ for } z = h + \zeta(x, t). \tag{3}$$

Here we have assumed that the potential energy is due to gravity and have taken the reference potential energy at the height $z = h$. In this form of Bernoulli's equation, we have absorbed some integration constants in the choice of time dependence of the velocity potential as discussed in your text (Eq. 54.11). The boundary conditions for this system take the form of zero vertical velocity at bottom of the tank:

$$\frac{\partial\Phi(x, 0, t)}{\partial z} = 0. \tag{4}$$

At the surface of the fluid, $z = h + \zeta(x, t)$ we expect that

$$v_z(x, z, t)\rfloor_{z=h+\zeta} = \frac{d\zeta}{dt} = \mathbf{v} \cdot \nabla\zeta + \frac{\partial\zeta}{\partial t}. \tag{5}$$

This relationship can be expressed as:

$$-\frac{\partial\Phi(x, z, t)}{\partial z} + \frac{\partial\Phi(x, z, t)}{\partial x}\frac{\partial\zeta(x, t)}{\partial x} - \frac{\partial\zeta(x, t)}{\partial t}\bigg\rfloor_{z=h+\zeta} = 0 \tag{6}$$

In this treatment, we assume seek the form of surface waves traveling along the $x-$ direction and assume that the effective wavelength is much larger than the height of the surface $h$. This allows us to approximate the $z-$ dependence of $\Phi(x, z, t)$ by means of a Taylor series expansion:

$$\Phi(x, z, t) \approx \Phi(x, 0, t) + z\frac{\partial\Phi}{\partial z}(x, 0, t) + \frac{z^2}{2}\frac{\partial^2\Phi}{\partial z^2}(x, 0, t) + \frac{z^3}{3!}\frac{\partial^3\Phi}{\partial z^3}(x, 0, t) + \frac{z^4}{4!}\frac{\partial^4\Phi}{\partial z^4}(x, 0, t)\cdots \tag{7}$$

This expansion can be simplified because of the bottom boundary condition (4) which ensures that all odd derivatives $\frac{\partial^n\Phi}{\partial z^n}(x, 0, t)$ vanish from the Taylor expansion (7). In addition, the Laplace equation

(2) allows us to convert all even derivatives with respect to $z$ to derivatives with respect to $x$. Therefore, the expansion (7) becomes:

$$\Phi(x,z,t) \approx \Phi(x,0,t) - \frac{z^2}{2}\frac{\partial^2\Phi}{\partial x^2}(x,0,t) + \frac{z^4}{4!}\frac{\partial^4\Phi}{\partial x^4}(x,0,t)\cdots \tag{8}$$

Before seeking the form of the nonlinear equations, we first consider the linearized version of these equations. We focus on the solution at the free surface $z = h + \zeta(x,t)$. The linear version of the Bernoulli equation evaluated at the free surface is

$$-\frac{\partial\Phi(x,h,t)}{\partial t} + g\zeta(x,t) = 0. \tag{9}$$

The linearized surface boundary condition is

$$-\frac{\partial\Phi(x,z,t)}{\partial z} - \frac{\partial\zeta(x,t)}{\partial t}\bigg]_{z=h+\zeta} = 0 \tag{10}$$

Using the Taylor's expansion in this surface boundary

$$-\frac{\partial\Phi(x,z,t)}{\partial z} \approx h\frac{\partial^2\Phi(x,0,t)}{\partial x^2} = \frac{\partial\zeta(x,t)}{\partial t}. \tag{11}$$

Eliminating $\zeta$ from the coupled Eqs. (9) and (11), we find

$$\frac{\partial^2\Phi(x,0,t)}{\partial t^2} = gh\frac{\partial^2\Phi(x,0,t)}{\partial x^2}, \tag{12}$$

a wave equation with velocity $c = \sqrt{gh}$.

We now return to treating the nonlinear equations.
For convenience we define $\phi(x,t) \equiv \Phi(x,0,t)$. Using Eq. (8), the Bernoulli equation (3) then becomes:

$$-\frac{\partial\phi}{\partial t} + \frac{(h+\zeta)^2}{2}\frac{\partial^3\phi}{\partial t\partial x^2} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left((h+\zeta)\frac{\partial^2\phi}{\partial x^2}\right)^2\right] + g\zeta = 0, \tag{13}$$

where we have discarded some of the higher order terms. Keeping all terms up to leading order in non-linearity and up to fourth order derivatives in the linear terms, the Bernoulli equation becomes:

$$-\frac{\partial\phi}{\partial t} + \frac{h^2}{2}\frac{\partial^3\phi}{\partial t\partial x^2} + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + g\zeta = 0. \tag{14}$$

Using a similar analysis and approximation, the surface definition equation (6) becomes:

$$\frac{\partial}{\partial x}\left((h+\zeta(x,t))\frac{\partial\phi}{\partial x}\right) - \frac{h^3}{3!}\frac{\partial^4\phi}{\partial x^4} - \frac{\partial\zeta}{\partial t} = 0, \tag{15}$$

We would like to solve Eqs. (14-15) for a traveling wave of the form:

$$\phi(x,t) = \chi(x-ct) \text{ and } \zeta(x,t) = \eta(x-ct), \tag{16}$$

where the speed of the wave $c$ will be determined. Letting $u \equiv x - ct$, Eqs. (14 becomes

$$c\frac{d\chi(u)}{du} - \frac{ch^2}{2}\frac{d^3\chi(u)}{du^3} + \frac{1}{2}\left(\frac{d\chi(u)}{du}\right)^2 + g\eta(u) = 0. \tag{17}$$

and 15) becomes

$$\frac{d}{du}\left((h+\eta(u))\frac{d\chi(u)}{du}\right) - \frac{h^3}{6}\frac{d^4\chi(u)}{du^4} + c\frac{d\eta(u)}{du} = 0. \tag{18}$$

The modified surface equation (18) can be integrated once with respect to u, choosing the constant of integration to be zero and giving the new form for the surface condition:

$$(h+\eta)\chi' - \frac{h^3}{6}\chi''' + c\eta = 0, \tag{19}$$

where we have abreviated derivatives with respect to $u$ with the "$'$" symbol. This equation, and the modified Bernoulli equation (17) which can be slightly rearranged

$$\chi' - \frac{h^2}{2}\chi''' + \frac{1}{2c}(\chi')^2 + \frac{g}{c}\eta = 0, \tag{20}$$

are now two coupled non-linear equations. In order to solve them, we make further approximations in terms of the expected magnitudes of the terms. Linear terms and lower derivatives are assumed to be larger than nonlinear terms and higher order derivatives. We also want to express the equations in terms of the surface function $\eta(u)$. The Bernoulli Equation (20) is thus approximated by

$$\chi' = -\frac{g}{c}\eta + \frac{h^2}{2}\chi''' - \frac{1}{2c}(\chi')^2 \approx -\frac{g}{c}\eta - \frac{h^2 g}{2c}\eta'' - \frac{g^2}{2c^3}\eta^2. \tag{21}$$

Using similar approximations, we can eliminate $\chi'(u)$ and its higher derivatives from the surface equation (19):

$$(h+\eta)\left(-\frac{g}{c}\eta - \frac{h^2 g}{2c}\eta'' - \frac{g^2}{2c^3}\eta^2\right) + \frac{h^3 g}{6c}\eta'' + c\eta = 0, \tag{22}$$

where some terms involving non-linearity of higher than 2 or involving higher order derivatives have been discarded. Collecting the leading terms, we obtain:

$$\left(1 - \frac{gh}{c^2}\right)\eta - \frac{gh^3}{3c^2}\eta'' - \frac{g}{c^2}\left(1 + \frac{gh}{2c^2}\right)\eta^2 = 0. \tag{23}$$

For the second two terms, *Fetter and Walecka* argue that it is consistent to approximate $gh \approx c^2$, which reduces (23) to

$$\left(1 - \frac{hg}{c^2}\right)\eta(u) - \frac{h^2}{3}\eta''(u) - \frac{3}{2h}\left[\eta(u)\right]^2 = 0. \tag{24}$$

Your text shows that a solution to Eq. (24) (corresponding to Eq. 56.30 of the text), with the initial condition $\eta(0) = \eta_0$ and $\eta'(0) = 0$, is the solitary wave form:

$$\zeta(x,t) = \eta(x-ct) = \eta_0 \operatorname{sech}^2\left(\sqrt{\frac{3\eta_0}{h}}\,\frac{x-ct}{2h}\right), \tag{25}$$

with
$$c = \sqrt{\frac{gh}{1 - \eta_0/h}} \approx \sqrt{gh}\left(1 + \frac{\eta_0}{2h}\right). \qquad (26)$$
The "standard" form of the related Korteweg-de Vries equation[2] is given in terms of the scaled variables $\bar{t}$ and $\bar{x}$ in terms of the function $\eta(\bar{x}, \bar{t})$ by
$$\frac{\partial \eta}{\partial \bar{t}} + 6\eta \frac{\partial \eta}{\partial \bar{x}} + \frac{\partial^3 \eta}{\partial \bar{x}^3} = 0, \qquad (27)$$
which has a solution
$$\eta(\bar{x}, \bar{t}) = \frac{\beta}{2}\,\mathrm{sech}^2\left[\frac{\sqrt{\beta}}{2}(\bar{x} - \beta\bar{t})\right]. \qquad (28)$$
This form is related to our results in the following way.
$$\beta = 2\eta_0, \qquad \bar{x} = \sqrt{\frac{3}{2h}}\frac{x}{h}, \qquad \text{and} \qquad \bar{t} = \sqrt{\frac{3}{2h}}\frac{ct}{2\eta_0 h}. \qquad (29)$$
To show how the reduced equation (24) is related to the Korteweg-de Vries equation, we first take the $u$ derivative to find:
$$\frac{\eta_0}{h}\eta' - \frac{h^2}{3}\eta''' - \frac{3}{h}\eta\eta' = 0, \qquad (30)$$
where we have used the relation
$$\frac{\eta_0}{h} = 1 - \frac{gh}{c^2}. \qquad (31)$$
Then we notice that
$$\frac{\partial \eta}{\partial t} = -c\frac{d\eta}{du} \qquad \text{and} \qquad \frac{\partial \eta}{\partial x} = \frac{d\eta}{du}, \qquad (32)$$
so that Eq. (30) can be written:
$$-\frac{\eta_0}{ch}\frac{\partial \eta}{\partial t} - \frac{h^2}{3}\frac{\partial^3 \eta}{\partial x^3} - \frac{3}{h}\eta\frac{\partial \eta}{\partial x} = 0. \qquad (33)$$
Substituting the transformation (29) into this partial differential equation yields the Korteweg-de Vries equation (27).

# References


[1] Alexander L. Fetter and John Dirk Walecka, **Theoretical Mechanics of Particles and Continua**, (McGraw Hill, 1980), Chapt. 10.

[2] Websites concerning solitons:

http://www.ma.hw.ac.uk/solitons/