

**PHY 711 Classical Mechanics and
Mathematical Methods**
10-10:50 AM MWF Olin 103

Plan for Lecture 10:

Continue reading Chapter 3 & 6

1. Constants of the motion

2. Conserved quantities

9/17/2014

PHY 711 Fall 2014 -- Lecture 10

1

Course schedule

(Preliminary schedule -- subject to frequent adjustment.) m>

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/27/2014	Chap. 1	Review of basic principles	#1
2	Fri, 8/29/2014	Chap. 1	Scattering theory	#2
3	Mon, 9/01/2014	Chap. 1	Scattering theory continued	#3
4	Wed, 9/03/2014	Chap. 2	Accelerated coordinate systems	#4
5	Fri, 9/05/2014	Chap. 3	Calculus of variations	#5
6	Mon, 9/08/2014	Chap. 3	Calculus of variations	#6
7	Wed, 9/10/2014	Chap. 3	Hamilton's principle	#7
8	Fri, 9/12/2014	Chap. 3 & 6	Hamilton's principle	#8
9	Mon, 9/15/2014	Chap. 3 & 6	Lagrangians with constraints	#9
10	Wed, 9/17/2014	Chap. 3 & 6	Lagrangians and constants of motion	#10
11	Fri, 9/19/2014			

9/17/2014

PHY 711 Fall 2014 -- Lecture 10

2

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Department of Physics

News



Andrea Belanger Awarded
Poster Prize



Ryan Godwin Awarded
Predoctoral Fellowship



Department Recognizes 2014

Events

Wed. Sept. 17, 2014
Physics Colloquium:
Dark energy and matter:
Prof. Enckcek, UNC-CH
Olin 101 4:00 PM
Refreshments at 3:30 PM
Olin Lobby

Sat. Sept. 20, 2014
Annual Alumni Reception
Winston Hall at 10 AM
Family friendly demo show
and reception -- a joint
venture of the Physics,
Biology, and Chemistry
Departments, welcoming all
alums and their families.

9/17/2014

PHY 711 Fall 2014 -- Lecture 10

3

WFU Physics Colloquium

TITLE: The Dark Sector's First Minute

SPEAKER: [Professor Adrienne L. Erickcek](#),

Department of Physics and Astronomy,
University of North Carolina at Chapel Hill

TIME: Wednesday September 17, 2014 at 4:00 PM

PLACE: Room 101 Olin Physical Laboratory

Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

ABSTRACT

There are two unknown actors on the cosmic stage: the dark energy that is accelerating the Universe's expansion and the dark matter that dominates galaxies and clusters. I will demonstrate how we can gain powerful insights into the nature of dark energy and dark matter by investigating their behavior in the most energetic environment ever realized: the early Universe. First, I will show that the early Universe exposes a critical flaw in chameleon gravity, which is a modified gravity theory that provides dark energy while mimicking general relativity within the Solar System. Second, I will describe how the abundance of dark matter microhalos probes the early Universe's own mystery: the origin of the initial density fluctuations that seeded cosmic structures. Finally, I will show that microhalos also provide a new window on the dynamics of the extremely early Universe and the origins of dark matter.

9/17/2014

PHY 711 Fall 2014 -- Lecture 10

4

Summary of Lagrangian formalism (without constraints)

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Note that if $\frac{\partial L}{\partial q_\sigma} = 0$, then $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} = 0$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_\sigma} = (\text{constant})$$

9/17/2014

PHY 711 Fall 2014 -- Lecture 10

5

Examples of constants of the motion:

Example 1: one-dimensional potential:

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z)$$

$$\Rightarrow \frac{d}{dt} m\dot{x} = 0 \quad \Rightarrow m\dot{x} \equiv p_x \text{ (constant)}$$

$$\Rightarrow \frac{d}{dt} m\dot{y} = 0 \quad \Rightarrow m\dot{y} \equiv p_y \text{ (constant)}$$

$$\Rightarrow \frac{d}{dt} m\dot{z} = -\frac{\partial V}{\partial z}$$

9/17/2014

PHY 711 Fall 2014 -- Lecture 10

6

Examples of constants of the motion:

Example 2: Motion in a central potential

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$

$$\Rightarrow \frac{d}{dt}mr^2\dot{\phi} = 0 \quad \Rightarrow mr^2\dot{\phi} \equiv p_{\phi} \text{ (constant)}$$

$$\Rightarrow \frac{d}{dt}mr\dot{r} = mr\dot{\phi}^2 - \frac{\partial V}{\partial r} = \frac{p_{\phi}^2}{mr^3} - \frac{\partial V}{\partial r}$$

9/17/2014

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7

Recall alternative form of Euler-Lagrange equations:

Starting from:

$$L = L(\{q_{\sigma}(t)\}, \{\dot{q}_{\sigma}(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_{\sigma}} - \frac{\partial L}{\partial q_{\sigma}} = 0$$

Also note that: $\frac{dL}{dt} = \sum_{\sigma} \frac{\partial L}{\partial q_{\sigma}} \dot{q}_{\sigma} + \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \ddot{q}_{\sigma} + \frac{\partial L}{\partial t}$

$$= \frac{d}{dt} \left(\sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} \right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt} \left(L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} \right) = \frac{\partial L}{\partial t}$$

9/17/2014

PHY 711 Fall 2014 -- Lecture 10

8

Additional constant of the motion:

If $\frac{\partial L}{\partial t} = 0$,

then: $\frac{d}{dt} \left(L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} \right) = \frac{\partial L}{\partial t} = 0$

$$\Rightarrow L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} = -E \text{ (constant)}$$

Example 1: one-dimensional potential:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z)$$

$$\Rightarrow \frac{d}{dt} \left(\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z) - m\dot{x}^2 - m\dot{y}^2 - m\dot{z}^2 \right) = 0$$

$$\Rightarrow -\left(\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(z) \right) = -E \text{ (constant)}$$

For this case, we also have $m\dot{x} \equiv p_x$ and $m\dot{y} \equiv p_y$

$$\Rightarrow E = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\dot{z}^2 + V(z)$$

9/17/2014

PHY 711 Fall 2014 -- Lecture 10

9

Additional constant of the motion -- continued:

If $\frac{\partial L}{\partial t} = 0$;

$$\text{then : } \frac{d}{dt} \left(L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} \right) = \frac{\partial L}{\partial t} = 0$$

$$\Rightarrow L - \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} = -E \quad (\text{constant})$$

Example 2 : Motion in a central potential

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r)$$

$$\Rightarrow \frac{d}{dt} \left(\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r) - m\dot{r}^2 - m r^2 \dot{\phi}^2 \right) = 0$$

$$\Rightarrow - \left(\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + V(r) \right) = -E \quad (\text{constant})$$

For this case, we also have $m r^2 \dot{\phi} \equiv p_{\phi}$

$$\Rightarrow E = \frac{p_{\phi}^2}{2mr^2} + \frac{1}{2} m \dot{r}^2 + V(r)$$

9/17/2014

PHY 711 Fall 2014 -- Lecture 10

10

Other examples

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + \dot{y}x)$$

$$\frac{\partial L}{\partial z} = 0 \quad \Rightarrow m\dot{z} = p_z \quad (\text{constant})$$

$$E = \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} - L$$

$$= m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + \dot{y}x)$$

$$- \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{2c} B_0 (-\dot{x}y + \dot{y}x)$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{p_z^2}{2m}$$

9/17/2014

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11

Other examples

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{c} B_0 \dot{x}y$$

$$\frac{\partial L}{\partial z} = 0 \quad \Rightarrow m\dot{z} = p_z \quad (\text{constant})$$

$$\frac{\partial L}{\partial x} = 0 \quad \Rightarrow m\dot{x} = p_x \quad (\text{constant})$$

$$E = \sum_{\sigma} \frac{\partial L}{\partial \dot{q}_{\sigma}} \dot{q}_{\sigma} - L$$

$$= m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{c} B_0 \dot{x}y$$

$$- \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{c} B_0 \dot{x}y$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m \dot{y}^2 + \frac{p_x^2}{2m} + \frac{p_z^2}{2m}$$

9/17/2014

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12

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

9/17/2014

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13

Mathematical transformations for continuous functions of several variables & Legendre transforms:

$$z(x, y) \Leftrightarrow x(y, z) ???$$

$$z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x} \right)_y dx + \left(\frac{\partial z}{\partial y} \right)_x dy$$

$$x(y, z) \Rightarrow dx = \left(\frac{\partial x}{\partial y} \right)_z dy + \left(\frac{\partial x}{\partial z} \right)_y dz$$

$$\text{But: } \left(\frac{\partial x}{\partial y} \right)_z = - \frac{(\partial z / \partial y)_x}{(\partial z / \partial x)_y}$$

9/17/2014

PHY 711 Fall 2014 – Lecture 10

14

Mathematical transformations for continuous functions of several variables & Legendre transforms continued:

$$z(x, y) \Rightarrow dz = \left(\frac{\partial z}{\partial x} \right)_y dx + \left(\frac{\partial z}{\partial y} \right)_x dy$$

$$\text{Let } u \equiv \left(\frac{\partial z}{\partial x} \right)_y \text{ and } v \equiv \left(\frac{\partial z}{\partial y} \right)_x$$

Define new function

$$w(u, y) \Rightarrow dw = \left(\frac{\partial w}{\partial u} \right)_y du + \left(\frac{\partial w}{\partial y} \right)_u dy$$

$$\text{For } w = z - ux, \quad dw = dz - udx - xdu = \cancel{u}dx + vdy - \cancel{u}dx - xdu$$

$$dw = -xdu + vdy$$

$$\Rightarrow \left(\frac{\partial w}{\partial u} \right)_y = -x \quad \left(\frac{\partial w}{\partial y} \right)_u = \left(\frac{\partial z}{\partial y} \right)_x = v$$

9/17/2014

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15

For thermodynamic functions:

Internal energy: $U = U(S, V)$

$$dU = TdS - PdV$$

$$dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV$$

$$\Rightarrow T = \left(\frac{\partial U}{\partial S}\right)_V \quad P = -\left(\frac{\partial U}{\partial V}\right)_S$$

Enthalpy: $H = H(S, P) = U + PV$

$$dH = dU + PdV + VdP = TdS + VdP = \left(\frac{\partial H}{\partial S}\right)_P dS + \left(\frac{\partial H}{\partial P}\right)_S dP$$

$$\Rightarrow T = \left(\frac{\partial H}{\partial S}\right)_P \quad V = \left(\frac{\partial H}{\partial P}\right)_S$$

9/17/2014

PHY 711 Fall 2014 -- Lecture 10

16

Name	Potential	Differential Form
Internal energy	$E(S, V, N)$	$dE = TdS - PdV + \mu dN$
Entropy	$S(E, V, N)$	$dS = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN$
Enthalpy	$H(S, P, N) = E + PV$	$dH = TdS + VdP + \mu dN$
Helmholtz free energy	$F(T, V, N) = E - TS$	$dF = -SdT - PdV + \mu dN$
Gibbs free energy	$G(T, P, N) = F + PV$	$dG = -SdT + VdP + \mu dN$
Landau potential	$\Omega(T, V, \mu) = F - \mu N$	$d\Omega = -SdT - PdV - N d\mu$

9/17/2014

PHY 711 Fall 2014 -- Lecture 10

17

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \quad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

9/17/2014

PHY 711 Fall 2014 -- Lecture 10

18

Hamiltonian picture – continued

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \quad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \quad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

9/17/2014

PHY 711 Fall 2014 – Lecture 10

19