

**PHY 711 Classical Mechanics and
Mathematical Methods**
10-10:50 AM MWF Olin 103

Plan for Lecture 15:

Continue reading Chapter 4

**1. Normal modes for extended one-
dimensional systems**

**2. Normal modes for 2 and 3
dimensional systems**

9/29/2014

PHY 711 Fall 2014 -- Lecture 15

1

Course schedule

(Preliminary schedule -- subject to frequent adjustment) rrr>

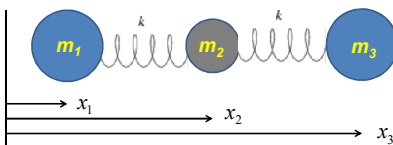
Date	F&W Reading	Topic	Assignment
1 Wed, 8/27/2014	Chap. 1	Review of basic principles	#1
2 Fri, 8/29/2014	Chap. 1	Scattering theory	#2
3 Mon, 9/01/2014	Chap. 1	Scattering theory continued	#3
4 Wed, 9/03/2014	Chap. 2	Accelerated coordinate systems	#4
5 Fri, 9/05/2014	Chap. 3	Calculus of variations	#5
6 Mon, 9/08/2014	Chap. 3	Calculus of variations	#6
7 Wed, 9/10/2014	Chap. 3	Hamilton's principle	#7
8 Fri, 9/12/2014	Chap. 3 & 6	Hamilton's principle	#8
9 Mon, 9/15/2014	Chap. 3 & 6	Lagrangians with constraints	#9
10 Wed, 9/17/2014	Chap. 3 & 6	Lagrangians and constants of motion	#10
11 Fri, 9/19/2014	Chap. 3 & 6	Hamiltonian formalism	#11
12 Mon, 9/22/2014	Chap. 3 & 6	Hamiltonian formalism	#11
13 Wed, 9/24/2014	Chap. 3 & 6	Hamiltonian Jacobi transformations	
14 Fri, 9/26/2014	Chap. 4	Small oscillations	Begin Take-Home
15 Mon, 9/29/2014	Chap. 4	Normal modes of motion	Continue Take-Home
16 Wed, 10/01/2014	Chap. 4	Normal modes of motion	Continue Take-Home
17 Fri, 10/03/2014	Chap. 4	Normal modes of motion	Take-Home due

9/29/2014

PHY 711 Fall 2014 -- Lecture 15

2

Example – linear molecule



$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2 - \frac{1}{2} k (x_2 - x_1 - \ell_{12})^2 - \frac{1}{2} k (x_3 - x_2 - \ell_{23})^2$$

9/29/2014

PHY 711 Fall 2014 -- Lecture 15

3

Coupled equations of motion :

$$m_1 \ddot{x}_1 = k(x_2 - x_1)$$

$$m_2 \ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2) = k(x_1 - 2x_2 + x_3)$$

$$m_3 \ddot{x}_3 = -k(x_3 - x_2)$$

Let $x_i(t) = X_i^\alpha e^{-i\omega_\alpha t}$

$$-\omega_\alpha^2 m_1 X_1^\alpha = k(X_2^\alpha - X_1^\alpha)$$

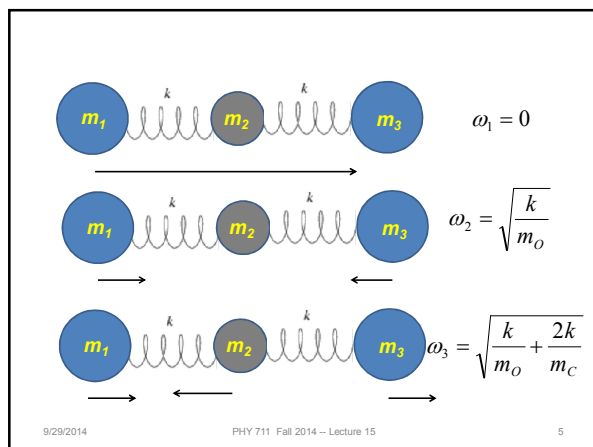
$$-\omega_\alpha^2 m_2 X_2^\alpha = k(X_1^\alpha - 2X_2^\alpha + X_3^\alpha)$$

$$-\omega_\alpha^2 m_3 X_3^\alpha = -k(X_3^\alpha - X_2^\alpha)$$

9/29/2014

PHY 711 Fall 2014 -- Lecture 15

4

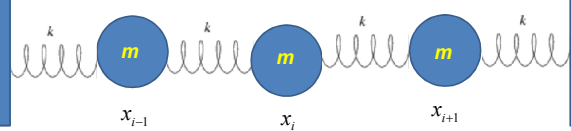


9/29/2014

PHY 711 Fall 2014 -- Lecture 15

5

Consider an extended system of masses and springs:



Note: each mass coordinate is measured relative to its equilibrium position x_i^0

$$L = T - V = \frac{1}{2} m \sum_{i=1}^N \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^N (x_{i+1} - x_i)^2$$

Note: In fact, we have N masses; x_0 and x_{N+1} will be treated using boundary conditions.

9/29/2014

PHY 711 Fall 2014 -- Lecture 15

6

$$L = T - V = \frac{1}{2} m \sum_{i=1}^N \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^N (x_{i+1} - x_i)^2$$

$$x_0 \equiv 0 \text{ and } x_{N+1} \equiv 0$$

From Euler - Lagrange equations :

$$m\ddot{x}_1 = k(x_2 - 2x_1)$$

$$m\ddot{x}_2 = k(x_3 - 2x_2 + x_1)$$

.....

$$m\ddot{x}_i = k(x_{i+1} - 2x_i + x_{i-1})$$

.....

$$m\ddot{x}_N = k(x_{N-1} - 2x_N)$$

9/29/2014

PHY 711 Fall 2014 -- Lecture 15

7

From Euler - Lagrange equations :

$$m\ddot{x}_j = k(x_{j+1} - 2x_j + x_{j-1}) \quad \text{with } x_0 = 0 = x_{N+1}$$

$$\text{Try: } x_j(t) = Ae^{-i\omega t + iqa j}$$

$$-\omega^2 Ae^{-i\omega t + iqa j} = \frac{k}{m} (e^{iqa} - 2 + e^{-iqa}) Ae^{-i\omega t + iqa j}$$

$$-\omega^2 = \frac{k}{m} (2\cos(qa) - 2)$$

$$\Rightarrow \omega^2 = \frac{4k}{m} \sin^2\left(\frac{qa}{2}\right)$$

9/29/2014

PHY 711 Fall 2014 -- Lecture 15

8

From Euler - Lagrange equations -- continued :

$$m\ddot{x}_j = k(x_{j+1} - 2x_j + x_{j-1}) \quad \text{with } x_0 = 0 = x_{N+1}$$

$$\text{Try: } x_j(t) = Ae^{-i\omega t + iqa j} \quad \Rightarrow \omega^2 = \frac{4k}{m} \sin^2\left(\frac{qa}{2}\right)$$

$$\text{Note that: } x_j(t) = Be^{-i\omega t - iqa j} \quad \Rightarrow \omega^2 = \frac{4k}{m} \sin^2\left(\frac{qa}{2}\right)$$

General solution :

$$x_j(t) = \Re(Ae^{-i\omega t + iqa j} + Be^{-i\omega t - iqa j})$$

Impose boundary conditions :

$$x_0(t) = \Re(Ae^{-i\omega t} + Be^{-i\omega t}) = 0$$

$$x_{N+1}(t) = \Re(Ae^{-i\omega t + iqa(N+1)} + Be^{-i\omega t - iqa(N+1)}) = 0$$

9/29/2014

PHY 711 Fall 2014 -- Lecture 15

9

Impose boundary conditions -- continued :

$$\begin{aligned}
 x_0(t) &= \Re(Ae^{-i\alpha t} + Be^{-i\alpha t}) = 0 \\
 x_{N+1}(t) &= \Re(Ae^{-i\alpha t + iqa(N+1)} + Be^{-i\alpha t - iqa(N+1)}) = 0 \\
 \Rightarrow B &= -A \\
 x_{N+1}(t) &= \Re(Ae^{-i\alpha t} (e^{iqa(N+1)} - e^{-iqa(N+1)})) = 0 \\
 \Rightarrow \sin(qa(N+1)) &= 0 \\
 \Rightarrow qa(N+1) &= \nu\pi \quad \text{where } \nu = 0, 1, 2, \dots \\
 qa &= \frac{\nu\pi}{N+1}
 \end{aligned}$$

9/29/2014

PHY 711 Fall 2014 -- Lecture 15

10

Recap -- solution for integer parameter ν

$$\begin{aligned}
 x_j(t) &= \Re\left(2iAe^{-i\omega_\nu t} \sin\left(\frac{\nu\pi j}{N+1}\right)\right) \\
 \omega_\nu^2 &= \frac{4k}{m} \sin^2\left(\frac{\nu\pi}{2(N+1)}\right)
 \end{aligned}$$

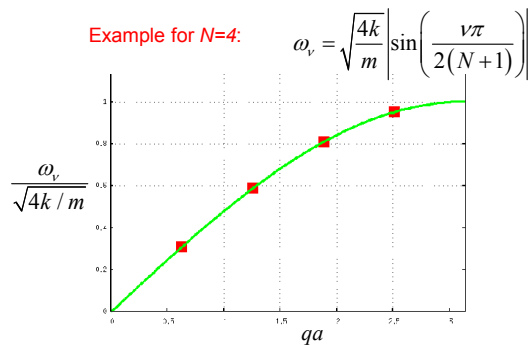
Note that non-trivial, unique values are
 $\nu = 1, 2, \dots, N$

9/29/2014

PHY 711 Fall 2014 -- Lecture 15

11

Example for $N=4$:



Note that solution form remains correct for $N \rightarrow \infty$

$$\omega(qa) = \sqrt{4k/m} \left| \sin(qa) \right|$$

9/29/2014

PHY 711 Fall 2014 -- Lecture 15

12

Consider an infinite system of masses and springs now with two kinds of masses:



Note: each mass coordinate is measured relative to its equilibrium position $x_i^0, y_i^0, \dots$

$$L = T - V$$

$$= \frac{1}{2} m \sum_{i=0}^{\infty} \dot{x}_i^2 + \frac{1}{2} M \sum_{i=0}^{\infty} \dot{y}_i^2 - \frac{1}{2} k \sum_{i=0}^{\infty} (x_{i+1} - y_i)^2 - \frac{1}{2} k \sum_{i=0}^{\infty} (y_i - x_i)^2$$

9/29/2014

PHY 711 Fall 2014 -- Lecture 15

13

$$L = T - V$$

$$= \frac{1}{2} m \sum_{i=0}^{\infty} \dot{x}_i^2 + \frac{1}{2} M \sum_{i=0}^{\infty} \dot{y}_i^2 - \frac{1}{2} k \sum_{i=0}^{\infty} (x_{i+1} - y_i)^2 - \frac{1}{2} k \sum_{i=0}^{\infty} (y_i - x_i)^2$$

Euler - Lagrange equations:

$$m\ddot{x}_j = k(y_{j-1} - 2x_j + y_j)$$

$$M\ddot{y}_j = k(x_j - 2y_j + x_{j+1})$$

Trial solution:

$$x_j(t) = A e^{-i\omega t + i2qaj}$$

$$y_j(t) = B e^{-i\omega t + i2qaj}$$

$$\begin{pmatrix} m\omega^2 - 2k & k(e^{-i2qa} + 1) \\ k(e^{i2qa} + 1) & M\omega^2 - 2k \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

9/29/2014

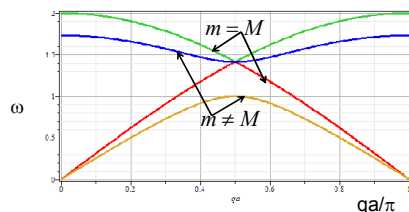
PHY 711 Fall 2014 -- Lecture 15

14

$$\begin{pmatrix} m\omega^2 - 2k & k(e^{-i2qa} + 1) \\ k(e^{i2qa} + 1) & M\omega^2 - 2k \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

Solutions:

$$\omega_{\pm}^2 = \frac{k}{m} + \frac{k}{M} \pm k \sqrt{\frac{1}{m^2} + \frac{1}{M^2} + \frac{2\cos(2qa)}{mM}}$$



9/29/2014

PHY 711 Fall 2014 -- Lecture 15

15

Eigenvectors:

For $qa = 0$:

$$\omega_- = 0 \quad \omega_+ = \sqrt{\frac{2k}{m} + \frac{2k}{M}}$$

$$\begin{pmatrix} A \\ B \end{pmatrix}_- = N \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} A \\ B \end{pmatrix}_+ = N \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $qa = \frac{\pi}{2}$:

$$\omega_- = \sqrt{\frac{2k}{M}} \quad \omega_+ = \sqrt{\frac{2k}{m}}$$

$$\begin{pmatrix} A \\ B \end{pmatrix}_- = N \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} A \\ B \end{pmatrix}_+ = N \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

9/29/2014

PHY 711 Fall 2014 -- Lecture 15

16
