

**PHY 711 Classical Mechanics and
Mathematical Methods**
10-10:50 AM MWF Olin 103

Plan for Lecture 17:

**Finish reading Chapter 4 and start reading
Chapter 7**

- 1. Linear versus non-linear oscillators**
- 2. Coupled motion of extended
systems; relationship to continuum
models & wave equation**

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Course schedule

(Preliminary schedule -- subject to frequent adjustment) rrr>

Date	F&W Reading	Topic	Assignment
1 Wed, 8/27/2014	Chap. 1	Review of basic principles	#1
2 Fri, 8/29/2014	Chap. 1	Scattering theory	#2
3 Mon, 9/01/2014	Chap. 1	Scattering theory continued	#3
4 Wed, 9/03/2014	Chap. 2	Accelerated coordinate systems	#4
5 Fri, 9/05/2014	Chap. 3	Calculus of variations	#5
6 Mon, 9/08/2014	Chap. 3	Calculus of variations	#6
7 Wed, 9/10/2014	Chap. 3	Hamilton's principle	#7
8 Fri, 9/12/2014	Chap. 3 & 6	Hamilton's principle	#8
9 Mon, 9/15/2014	Chap. 3 & 6	Lagrangians with constraints	#9
10 Wed, 9/17/2014	Chap. 3 & 6	Lagrangians and constants of motion	#10
11 Fri, 9/19/2014	Chap. 3 & 6	Hamiltonian formalism	#11
12 Mon, 9/22/2014	Chap. 3 & 6	Hamiltonian formalism	#11
13 Wed, 9/24/2014	Chap. 3 & 6	Hamiltonian Jacobi transformations	
14 Fri, 9/26/2014	Chap. 4	Small oscillations	Begin Take-Home
15 Mon, 9/29/2014	Chap. 4	Normal modes of motion	Continue Take-Home
16 Wed, 10/01/2014	Chap. 4	Normal modes of motion	Continue Take-Home
17 Fri, 10/03/2014	Chap. 4	Normal modes of motion	Take-Home due

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Linear oscillator equations (example from one dimension)

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left. \frac{d^2 V}{dx^2} \right|_{x_{eq}} + \dots$$

$$\Rightarrow \frac{1}{2} kx^2 \equiv \frac{1}{2} m\omega^2 x^2$$

$$L(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2$$

Euler - Lagrange equations :

$$\ddot{x} = -\omega^2 x$$

Superposition :

Suppose that the functions $x_1(t)$ and $x_2(t)$ are solutions

$$\Rightarrow Ax_1(t) + Bx_2(t) \text{ are also solutions (all } A, B)$$

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Non - linear oscillator equations (example from one dimension)

$$V(x) \approx V(x_{eq}) + \frac{1}{2}(x - x_{eq})^2 \left. \frac{d^2 V}{dx^2} \right|_{x_{eq}} + \frac{1}{4!}(x - x_{eq})^4 \left. \frac{d^4 V}{dx^4} \right|_{x_{eq}} + \dots$$

$$\Rightarrow \frac{1}{2} m \omega^2 \left(x^2 + \frac{1}{2} \epsilon x^4 \right)$$

$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 \left(x^2 + \frac{1}{2} \epsilon x^4 \right)$$

Euler - Lagrange equations :

$$\ddot{x} = -\omega^2 (x + \epsilon x^3)$$

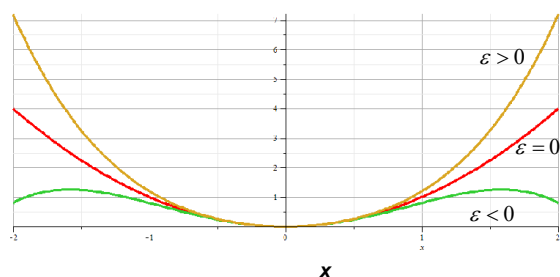
Superposition -- no longer applies

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$$V(x) \approx \frac{1}{2} m \omega^2 (x^2 + \epsilon x^4)$$



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Non - linear example -- continued

$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 \left(x^2 + \frac{1}{2} \epsilon x^4 \right)$$

Euler - Lagrange equations :

$$\ddot{x} + \omega^2 (x + \epsilon x^3) = 0$$

Perturbation expansion :

$$x(t) = x_0(t) + \epsilon x_1(t) + \dots$$

Euler - Lagrange equations :

$$\text{zero order : } \ddot{x}_0 + \omega^2 x_0 = 0$$

$$\text{first order : } \ddot{x}_1 + \omega^2 x_1 + \omega^2 x_0^3 = 0$$

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Non - linear example -- continued

$$\ddot{x} + \omega^2(x + \varepsilon x^3) = 0$$

Initial conditions :

Perturbation expansion : $x(0) = X_0 \quad \dot{x}(0) = 0$

$$x(t) = x_0(t) + \varepsilon x_1(t) + \dots$$

Euler - Lagrange equations :

zero order : $\ddot{x}_0 + \omega^2 x_0 = 0 \quad \Rightarrow x_0(t) = X_0 \cos(\omega t)$

first order : $\ddot{x}_1 + \omega^2 x_1 + \omega^2 x_0^3 = 0$

$$\Rightarrow \ddot{x}_1(t) + \omega^2 x_1(t) = -X_0^3 \cos^3(\omega t) = -\frac{X_0^3}{4} (3\cos(\omega t) + \cos(3\omega t))$$

$$\Rightarrow x_1(t) = -\frac{X_0^3}{8\omega^2} \left\{ 3\omega t \sin(\omega t) + \frac{1}{4} [\cos(\omega t) - \cos(3\omega t)] \right\}$$

$$x(t) = X_0 \cos(\omega t) - \varepsilon \frac{X_0^3}{8\omega^2} \left\{ 3\omega t \sin(\omega t) + \frac{1}{4} [\cos(\omega t) - \cos(3\omega t)] \right\} + O(\varepsilon^2)$$

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Non - linear example -- continued

$$\ddot{x} + \omega^2(x + \varepsilon x^3) = 0$$

Initial conditions :

$$x(0) = X_0 \quad \dot{x}(0) = 0$$

Perturbation expansion :

$$x(t) = x_0(t) + \varepsilon x_1(t) + \dots$$

Previous result (blows up at large t):

$$x(t) = X_0 \cos(\omega t) - \varepsilon \frac{X_0^3}{8\omega^2} \left\{ 3\omega t \sin(\omega t) + \frac{1}{4} [\cos(\omega t) - \cos(3\omega t)] \right\} + O(\varepsilon^2)$$

By rearranging terms (allowing effective frequency to vary):

$$x(t) = X_0 \cos \left(\omega \left(1 + \varepsilon \frac{3X_0^2}{8\omega} \right) t \right) - \varepsilon \frac{X_0^3}{32\omega^2} \{ \cos(\omega t) - \cos(3\omega t) \} + O(\varepsilon^2)$$

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Non - linear example with driving term -- Duffing equation

Georg Duffing ~ 1915

$$\ddot{x} + \omega^2(x + \varepsilon x^3) = A \cos(\Omega t)$$

Trial solution from: $x(t) \approx c_1 \cos(\Omega t) + c_3 \cos(3\Omega t)$

$$\left[(\omega^2 - \Omega^2) c_1 - \varepsilon \frac{3}{4} \omega^2 c_1^3 [1 + \dots] - A \right] \cos(\Omega t) +$$

$$\left[(\omega^2 - 9\Omega^2) c_3 - \varepsilon \frac{1}{4} \omega^2 c_1^3 [1 + \dots] \right] \cos(3\Omega t) + \dots = 0$$

Approximate solution: (assume $\frac{c_3}{c_1} \ll 1$)

$$\frac{c_3}{c_1} \approx \varepsilon \frac{1}{4} c_1^2 \frac{1}{1 - 9\omega^2 / \Omega^2}$$

$$(\omega^2 - \Omega^2) c_1 - \varepsilon \frac{3}{4} \omega^2 c_1^3 - A = 0$$

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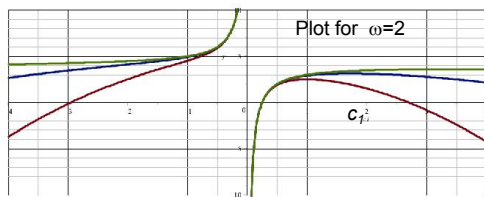
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Duffing oscillator -- continued

$$(\omega^2 - \Omega^2)c_1 - \varepsilon \frac{3}{4}\omega^2 c_1^3 - A = 0$$

$$\Omega^2 = \omega^2 - \varepsilon \frac{3}{4}\omega^2 c_1^2 - \frac{A}{c_1}$$



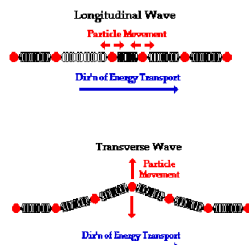
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Returning to linear case; continuum limit --
Longitudinal versus transverse vibrations
Images from web page:

<http://www.physicsclassroom.com/class/waves/u10l1c.cfm>

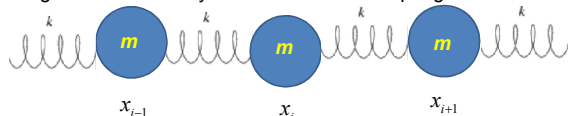


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Longitudinal case: a system of masses and springs:



$$L = T - V = \frac{1}{2}m \sum_{i=0}^{\infty} \dot{x}_i^2 - \frac{1}{2}k \sum_{i=0}^{\infty} (x_{i+1} - x_i)^2$$

$$\Rightarrow m\ddot{x}_i = k(x_{i+1} - 2x_i + x_{i-1})$$

Now imagine the continuum version of this system :

$$x_i(t) \Rightarrow \mu(x_i, t) \quad \ddot{x}_i \Rightarrow \frac{\partial^2 \mu}{\partial t^2}$$

$$x_{i+1} - 2x_i + x_{i-1} \Rightarrow \frac{\partial^2 \mu}{\partial x^2} (\Delta x)^2$$

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Discrete equation : $m\ddot{x}_i = k(x_{i+1} - 2x_i + x_{i-1}))$

Continuum equation : $m \frac{\partial^2 \mu}{\partial t^2} = k(\Delta x)^2 \frac{\partial^2 \mu}{\partial x^2}$

$$\frac{\partial^2 \mu}{\partial t^2} = \left(\frac{k\Delta x}{m/\Delta x} \right) \frac{\partial^2 \mu}{\partial x^2}$$

 system parameter with
units of (velocity)²

For transverse oscillations on a string
with tension τ and mass/length σ :

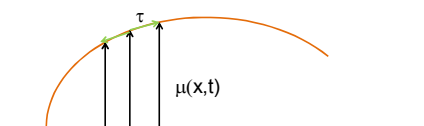
$$\left(\frac{k\Delta x}{m/\Delta x} \right) \Rightarrow \frac{\tau}{\sigma}$$

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Transverse displacement:



Wave equation :

$$\frac{\partial^2 \mu}{\partial t^2} = c^2 \frac{\partial^2 \mu}{\partial x^2}$$

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Lagrangian for continuous system :

Denote the generalized displacement by $\mu(x,t)$:

$$L = L\left(\mu, \frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial t}; x, t\right)$$

Hamilton's principle :

$$\delta \int_{t_i}^{t_f} dt \int_{x_i}^{x_f} dx L\left(\mu, \frac{\partial \mu}{\partial x}, \frac{\partial \mu}{\partial t}; x, t\right) = 0$$

$$\Rightarrow \frac{\partial L}{\partial \mu} - \frac{\partial}{\partial x} \frac{\partial L}{\partial (\partial \mu / \partial x)} - \frac{\partial}{\partial t} \frac{\partial L}{\partial (\partial \mu / \partial t)} = 0$$

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Euler - Lagrange equations for continuous system :

$$\frac{\partial L}{\partial \mu} - \frac{\partial}{\partial x} \frac{\partial L}{\partial (\partial \mu / \partial x)} - \frac{\partial}{\partial t} \frac{\partial L}{\partial (\partial \mu / \partial t)} = 0$$

Example :

$$L = \frac{\sigma}{2} \left(\frac{\partial \mu}{\partial t} \right)^2 - \frac{\tau}{2} \left(\frac{\partial \mu}{\partial x} \right)^2$$

$$\Rightarrow \sigma \frac{\partial^2 \mu}{\partial t^2} - \tau \frac{\partial^2 \mu}{\partial x^2} = 0$$

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{for } c^2 = \frac{\tau}{\sigma}$$

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General solutions $\mu(x, t)$ to the wave equation :

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x, t) = f(x - ct) + g(x + ct)$$

satisfies the wave equation.

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Initial value solutions $\mu(x, t)$ to the wave equation;
attributed to D' Alembert :

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x, 0) = \phi(x) \text{ and } \frac{\partial \mu}{\partial t}(x, 0) = \psi(x)$$

Assume :

$$\mu(x, t) = f(x - ct) + g(x + ct)$$

$$\text{then : } \mu(x, 0) = \phi(x) = f(x) + g(x)$$

$$\frac{\partial \mu}{\partial t}(x, 0) = \psi(x) = -c \left(\frac{df(x)}{dx} - \frac{dg(x)}{dx} \right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c} \int \psi(x') dx'$$

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Solution -- continued: $\mu(x,t) = f(x-ct) + g(x+ct)$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c \left(\frac{df(x)}{dx} - \frac{dg(x)}{dx} \right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c} \int \psi(x') dx'$$

For each x , find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2} \left(\phi(x) - \frac{1}{c} \int \psi(x') dx' \right)$$

$$g(x) = \frac{1}{2} \left(\phi(x) + \frac{1}{c} \int \psi(x') dx' \right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2} \left(\phi(x-ct) + \phi(x+ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x') dx'$$

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Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = e^{-x^2/\sigma^2} \text{ and } \frac{\partial \mu}{\partial t}(x,0) = 0$$

$$\Rightarrow \mu(x,t) = \frac{1}{2} \left(e^{-(x+ct)^2/\sigma^2} + e^{-(x-ct)^2/\sigma^2} \right)$$

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Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = 0 \text{ and } \frac{\partial \mu}{\partial t}(x,0) = -\frac{2x}{\sigma^2} e^{-x^2/\sigma^2}$$

$$\Rightarrow \mu(x,t) = \frac{1}{2c} \left(e^{-(x+ct)^2/\sigma^2} - e^{-(x-ct)^2/\sigma^2} \right)$$

Note that $\frac{\partial \mu(x,t)}{\partial t} = -\frac{1}{\sigma^2} \left((x+ct) e^{-(x+ct)^2/\sigma^2} + (x-ct) e^{-(x-ct)^2/\sigma^2} \right)$

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