PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 18:

Finish reading Chapter 4 and start reading Chapter 7

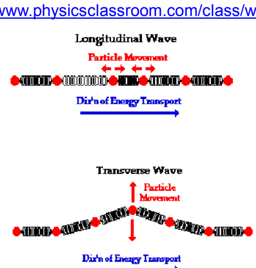
1. Coupled motion of extended systems; relationship to continuum models & wave equation
2. Analytic methods for solving Sturm-Liouville equations

Course schedule

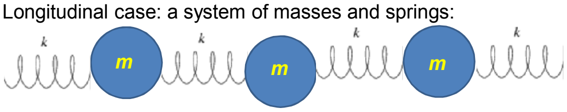
(Preliminary schedule -- subject to frequent adjustment.) rrr>

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/27/2014	Chap. 1	Review of basic principles	#1
2	Fri, 8/29/2014	Chap. 1	Scattering theory	#2
3	Mon, 9/01/2014	Chap. 1	Scattering theory continued	#3
4	Wed, 9/03/2014	Chap. 2	Accelerated coordinate systems	#4
5	Fri, 9/05/2014	Chap. 3	Calculus of variations	#5
6	Mon, 9/08/2014	Chap. 3	Calculus of variations	#6
7	Wed, 9/10/2014	Chap. 3	Hamilton's principle	#7
8	Fri, 9/12/2014	Chap. 3 & 6	Hamilton's principle	#8
9	Mon, 9/15/2014	Chap. 3 & 6	Lagrangians with constraints	#9
10	Wed, 9/17/2014	Chap. 3 & 6	Lagrangians and constants of motion	#10
11	Fri, 9/19/2014	Chap. 3 & 6	Hamiltonian formalism	#11
12	Mon, 9/22/2014	Chap. 3 & 6	Hamiltonian formalism	#11
13	Wed, 9/24/2014	Chap. 3 & 6	Hamiltonian Jacobi transformations	
14	Fri, 9/26/2014	Chap. 4	Small oscillations	Begin Take-Home
15	Mon, 9/29/2014	Chap. 4	Normal modes of motion	Continue Take-Home
16	Wed, 10/01/2014	Chap. 4	Normal modes of motion	Continue Take-Home
17	Fri, 10/03/2014	Chap. 4	Normal modes of motion	Take-Home due
18	Mon, 10/06/2014	Chap. 7	Wave motion	#12
19	Wed, 10/08/2014			

Returning to linear case; continuum limit --
Longitudinal versus transverse vibrations
Images from web page:
http://www.physicsclassroom.com/class/waves/u10l1c.cfm



Longitudinal case: a system of masses and springs:



x_{i-1} x_i x_{i+1}

$$L = T - V = \frac{1}{2} m \sum_{i=0}^{\infty} \dot{x}_i^2 - \frac{1}{2} k \sum_{i=0}^{\infty} \left(x_{i+1} - x_i\right)^2$$

$$\Rightarrow m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1}\right)$$

Now imagine the continuum version of this system :

$$x_i(t) \Rightarrow \mu(x_i, t) \qquad \ddot{x}_i \Rightarrow \frac{\partial^2 \mu}{\partial t^2}$$

$$x_{i+1} - 2x_i + x_{i-1} \Rightarrow \frac{\partial^2 \mu}{\partial x^2} (\Delta x)^2$$

Discrete equation : $m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1}\right)$

Continuum equation : $m \dfrac{\partial^2 \mu}{\partial t^2} = k(\Delta x)^2 \dfrac{\partial^2 \mu}{\partial x^2}$

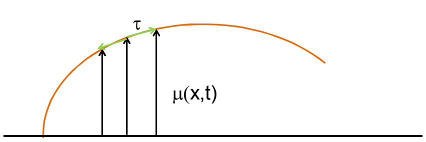
$$\frac{\partial^2 \mu}{\partial t^2} = \left(\frac{k\Delta x}{m / \Delta x}\right) \frac{\partial^2 \mu}{\partial x^2}$$

system parameter with units of (velocity)^2

For transverse oscillations on a string with tension τ and mass/length σ :

$$\left(\frac{k\Delta x}{m / \Delta x}\right) \Rightarrow \frac{\tau}{\sigma}$$

Transverse displacement:



Wave equation :

$$\frac{\partial^2 \mu}{\partial t^2} = c^2 \frac{\partial^2 \mu}{\partial x^2}$$

Lagrangian for continuous system :

Denote the generalized displacement by $\mu(x,t)$:

$$L = L\left(\mu, \frac{\partial\mu}{\partial x}, \frac{\partial\mu}{\partial t}; x, t\right)$$

Hamilton's principle :

$$\delta\int_{t_i}^{t_f} dt \int_{x_i}^{x_f} dx\, L\left(\mu, \frac{\partial\mu}{\partial x}, \frac{\partial\mu}{\partial t}; x, t\right) = 0$$

$$\Rightarrow \frac{\partial L}{\partial\mu} - \frac{\partial}{\partial x}\frac{\partial L}{\partial(\partial\mu/\partial x)} - \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial\mu/\partial t)} = 0$$

Euler - Lagrange equations for continuous system :

$$\frac{\partial L}{\partial\mu} - \frac{\partial}{\partial x}\frac{\partial L}{\partial(\partial\mu/\partial x)} - \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial\mu/\partial t)} = 0$$

Example :

$$L = \frac{\sigma}{2}\left(\frac{\partial\mu}{\partial t}\right)^2 - \frac{\tau}{2}\left(\frac{\partial\mu}{\partial x}\right)^2$$

$$\Rightarrow \sigma\frac{\partial^2\mu}{\partial t^2} - \tau\frac{\partial^2\mu}{\partial x^2} = 0$$

$$\frac{\partial^2\mu}{\partial t^2} - c^2\frac{\partial^2\mu}{\partial x^2} = 0 \quad \text{for} \quad c^2 = \frac{\tau}{\sigma}$$

General solutions $\mu(x,t)$ to the wave equation :

$$\frac{\partial^2\mu}{\partial t^2} - c^2\frac{\partial^2\mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

satisfies the wave equation.

Initial value solutions $\mu(x,t)$ to the wave equation;
attributed to D'Alembert:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = \phi(x) \text{ and } \frac{\partial \mu}{\partial t}(x,0) = \psi(x)$$

Assume:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

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Solution -- continued: $\mu(x,t) = f(x-ct) + g(x+ct)$

then: $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c\left(\frac{df(x)}{dx} - \frac{dg(x)}{dx}\right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int^{x} \psi(x')dx'$$

For each x, find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2}\left(\phi(x) - \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$g(x) = \frac{1}{2}\left(\phi(x) + \frac{1}{c}\int^{x} \psi(x')dx'\right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2}\left(\phi(x-ct) + \phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

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Example:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = e^{-x^2/\sigma^2} \text{ and } \frac{\partial \mu}{\partial t}(x,0) = 0$$

$$\Rightarrow \mu(\mathrm{x,t}) = \frac{1}{2}\left(e^{-(x+ct)^2/\sigma^2} + e^{-(x-ct)^2/\sigma^2}\right)$$

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Example:

$$\frac{\partial^2\mu}{\partial t^2} - c^2\frac{\partial^2\mu}{\partial x^2} = 0 \quad \text{where } \mu(x,0) = 0 \text{ and } \frac{\partial\mu}{\partial t}(x,0) = -\frac{2x}{\sigma^2}e^{-x^2/\sigma^2}$$

$$\Rightarrow \mu(x,t) = \frac{1}{2c}\left(e^{-(x+ct)^2/\sigma^2} - e^{-(x-ct)^2/\sigma^2}\right)$$

Note that $\frac{\partial\mu(x,t)}{\partial t} = -\frac{1}{\sigma^2}\left((x+ct)e^{-(x+ct)^2/\sigma^2} + (x-ct)e^{-(x-ct)^2/\sigma^2}\right)$

Other methods for solving the wave equation for $\mu(x,t)$:

$$\frac{\partial^2\mu}{\partial t^2} - c^2\frac{\partial^2\mu}{\partial x^2} = 0$$

Bernoulli's method:

Assume boundary values $\mu(0,t) = \mu(l,t) = 0$.

Let $\mu(x,t) = \rho(x)e^{-i\omega t}$

$$\Rightarrow \frac{d^2\rho(x)}{dx^2} + \frac{\omega^2}{c^2}\rho(x) = 0$$

Eigenvalue equation for $\rho_n(x)$:

$$\frac{d^2\rho_n(x)}{dx^2} = -\lambda_n\rho_n(x)$$

Bernoulli's method continued

$$\rho_n(x) = \sqrt{\frac{2}{l\sigma}}\sin\left(\frac{n\pi x}{l}\right) \qquad \lambda_n = \left(\frac{n\pi}{l}\right)^2 = \frac{\omega_n^2}{c^2}$$

Normalization of $\rho_n(x)$:

$$\int_0^l \rho_n(x)\rho_m(x)\sigma(x)dx = \delta_{nm}$$

General solution of wave equation (for these boundary values):

$$\mu(x,t) = \Re\left(\sum_n C_n\rho_n(x)e^{-i\omega_n t}\right)$$

Initial value solutions $\mu(x,t)$ to the wave equation using Brillouin method:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0 \qquad \text{where } \mu(x,0) = \varphi(x) \text{ and } \frac{\partial \mu}{\partial t}(x,0) = \psi(x)$$

$$\mu(x,t) = \Re\left(\sum_n C_n \rho_n(x) e^{-i\omega_n t} \right)$$

$$\mu(x,0) = \Re\left(\sum_n C_n \rho_n(x) \right) = \varphi(x)$$

$$\frac{\partial \mu(x,0)}{\partial t} = \Re\left(-i \sum_n C_n \omega_n \rho_n(x) \right) = \psi(x)$$

Initial value solutions $\mu(x,t)$ to the wave equation using Brillouin method -- continued:

Further assuming coefficients C_n to be real and $\varphi(x)$ to be defined in the domain $0 \leq x \leq l$:

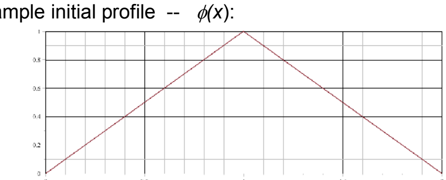
$$\Rightarrow C_n = \int_0^l \rho_n(x)\varphi(x)\sigma(x)dx$$

$$\mu(x,t) = \Re\left(\sum_n C_n \rho_n(x) e^{-i\omega_n t} \right)$$

Note that accuracy of solution depends on accuracy of initial condition:

$$\mu(x,0) = \Re\left(\sum_n C_n \rho_n(x) \right) = \varphi(x)$$

Example initial profile -- $\phi(x)$:



Four term expansion with $\rho_n(x) = \sin\left(\frac{n\pi x}{2}\right)$