PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 20:

Summary of mathematical methods

1. **Sturm-Liouville equations**
2. **Green's function methods**
3. **Laplace transform**
4. **Contour integration**

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/27/2014	Chap. 1	Review of basic principles	#1
2	Fri, 8/29/2014	Chap. 1	Scattering theory	#2
3	Mon, 9/01/2014	Chap. 1	Scattering theory continued	#3
4	Wed, 9/03/2014	Chap. 2	Accelerated coordinate systems	#4
5	Fri, 9/05/2014	Chap. 3	Calculus of variations	#5
6	Mon, 9/08/2014	Chap. 3	Calculus of variations	#6
7	Wed, 9/10/2014	Chap. 3	Hamilton's principle	#7
8	Fri, 9/12/2014	Chap. 3 & 6	Hamilton's principle	#8
9	Mon, 9/15/2014	Chap. 3 & 6	Lagrangians with constraints	#9
10	Wed, 9/17/2014	Chap. 3 & 6	Lagrangians and constants of motion	#10
11	Fri, 9/19/2014	Chap. 3 & 6	Hamiltonian formalism	#11
12	Mon, 9/22/2014	Chap. 3 & 6	Hamiltonian formalism	#11
13	Wed, 9/24/2014	Chap. 3 & 6	Hamiltonian Jacobi transformations	
14	Fri, 9/26/2014	Chap. 4	Small oscillations	Begin Take-Home
15	Mon, 9/29/2014	Chap. 4	Normal modes of motion	Continue Take-Home
16	Wed, 10/01/2014	Chap. 4	Normal modes of motion	Continue Take-Home
17	Fri, 10/03/2014	Chap. 4	Normal modes of motion	Take-Home due
18	Mon, 10/06/2014	Chap. 7	Wave motion	#12
19	Wed, 10/08/2014	Chap. 7	Sturm-Liouville Equations	#13
20	Fri, 10/10/2014	Chap. 7	Sturm-Liouville Equations	#13

Sturm-Liouville equation (assume all functions and constants are real):

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi_0(x)=0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi(x)=F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

We can prove as a general property of the Sturm-Liouville system, the eigenfunctions $f_n(x)$ are orthogonal

$$\int_a^b \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n,$$

$$\text{where } N_n \equiv \int_a^b \sigma(x)(f_n(x))^2 dx.$$

It can be shown that for any reasonable function *h(x)*, defined within the interval $a < x < b$, we can expand that function as a linear combination of the eigenfunctions $f_n(x)$

$$h(x) \approx \sum_n C_n f_n(x),$$

$$\text{where} \quad C_n = \frac{1}{N_n}\int_a^b \sigma(x')h(x')f_n(x')dx'.$$

These ideas lead to the notion that the set of eigenfunctions $f_n(x)$ form a ``complete'' set in the sense of ``spanning'' the space of all functions in the interval $a < x < b$, as summarized by the statement:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta(x-x').$$

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In general, there are several techniques to determine the eigenvalues λ_n and eigenfunctions $f_n(x)$. When it is not possible to find the ``exact'' functions, there are several powerful approximation techniques. For example, the lowest eigenvalue can be approximated by minimizing the function

$$\lambda_0 \le \frac{\langle \tilde{h}|S|\tilde{h}\rangle}{\langle \tilde{h}|\sigma|\tilde{h}\rangle},$$

where $\tilde{h}(x)$ is a variable function which satisfies the correct boundary values. The ``proof'' of this inequality is based on the notion that $\tilde{h}(x)$ can in principle be expanded in terms of the (unknown) exact eigenfunctions $f_n(x)$: $\tilde{h}(x) = \sum_n C_n f_n(x)$, where the coefficients C_n can be assumed to be real.

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Estimation of the lowest eigenvalue – continued:

From the eigenfunction equation, we know that

$$S(x)\tilde{h}(x) = S(x)\sum_n C_n f_n(x) = \sum_n C_n \lambda_n \sigma(x) f_n(x).$$

It follows that:

$$\langle \tilde{h}|S|\tilde{h}\rangle = \int_a^b \tilde{h}(x)S(x)\tilde{h}(x)dx = \sum_n |C_n|^2 N_n \lambda_n.$$

It also follows that:

$$\langle \tilde{h}|\sigma|\tilde{h}\rangle = \int_a^b \tilde{h}(x)\sigma(x)\tilde{h}(x)dx = \sum_n |C_n|^2 N_n,$$

$$\text{Therefore} \quad \frac{\langle \tilde{h}|S|\tilde{h}\rangle}{\langle \tilde{h}|\sigma|\tilde{h}\rangle} = \frac{\sum_n |C_n|^2 N_n \lambda_n}{\sum_n |C_n|^2 N_n} \ge \lambda_0.$$

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Rayleigh-Ritz method of estimating the lowest eigenvalue

$$\lambda_0 \le \frac{\langle \tilde{h} | S | \tilde{h} \rangle}{\langle \tilde{h} | \sigma | \tilde{h} \rangle},$$

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Oct. 7, 2014

Continue reading Chapter 7 in **Fetter & Walecka** and the lecture notes.

1. Consider the eigenvalue problem for $u(x)$ over the interval $0 \le x \le a$ such that $u(0)=u(a)=0$: $d^2u_n/dx^2 = -\lambda_n u_n(x)$.
   a. Find the smallest eigenvalue λ_1.
   b. Use the Rayleigh-Ritz approximation to estimate λ_1 with one or more trial functions (such as $u(x)=x(a-x)$).

Recap: Sturm-Liouville equation:

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x)\right)\phi_0(x) = 0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x)\right)\phi(x) = F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x)\right)f_n(x) = \lambda_n\sigma(x)f_n(x)$$

Example: $\tau(x)=1;\quad \sigma(x)=1;\quad v(x)=0;\quad a=0$ and $b=L$

$\lambda=1;\quad F(x)=F_0\sin\left(\frac{\pi x}{L}\right)$

Inhomogenous equation:

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x) = F_0\sin\left(\frac{\pi x}{L}\right)$$

Eigenvalue equation:

$$\left(-\frac{d^2}{dx^2}\right)f_n(x) = \lambda_n f_n(x)$$

Eigenfunctions: $f_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$

Eigenvalues: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

Completeness of eigenfunctions:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta(x-x')$$

In this example: $\frac{2}{L}\sum_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right) = \delta(x-x')$

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Green's function for the example :

$$G(x,x')=\sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}=\frac{2}{L}\sum_n\frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}$$

Using Green's function to solve inhomogenous equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

$$\phi(x)=\phi_0(x)+\int_0^L G(x,x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$=\phi_0(x)+\frac{2}{L}\sum_n\left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}\int_0^L\sin\left(\frac{n\pi x'}{L}\right)F_0\sin\left(\frac{\pi x'}{L}\right)dx'\right]$$

$$=\phi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

Alternate Green's function method :

$$G(x,x')=\frac{1}{W}g_a(x_<)g_B(x_>)$$

$$\left(-\frac{d^2}{dx^2}-1\right)g_i(x)=0 \qquad \Rightarrow g_a(x)=\sin(x);\quad g_b(x)=\sin(L-x);$$

$$W=g_b(x)\frac{dg_a(x)}{dx}-g_a(x)\frac{dg_b(x)}{dx}=\sin(L-x)\cos(x)+\sin(x)\cos(L-x)$$
$$=\sin(L)$$

$$\phi(x)=\phi_0(x)+\frac{\sin(L-x)}{\sin(L)}\int_0^x\sin(x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$
$$+\frac{\sin(x)}{\sin(L)}\int_x^L\sin(L-x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$\phi(x)=\phi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$

Laplace transforms

Laplace transforms can be used to solve initial value problems. The Laplace transform of a function $\phi(x)$ is defined as

$$\mathcal{L}_\phi(p) \equiv \int_0^\infty e^{-px}\phi(x)dx. \tag{24}$$

Assuming that $\phi(x)$ is well-behaved in the interval $0 \leq x \leq \infty$, the following properties are useful:

$$\mathcal{L}_{d\phi/dx}(p) = -\phi(0) + p\mathcal{L}_\phi(p), \tag{25}$$

and

$$\mathcal{L}_{d^2\phi/dx^2}(p) = -\frac{d\phi(0)}{dx} - p\phi(0) + p^2\mathcal{L}_\phi(p). \tag{26}$$

These identities allow us to turn a differential equation for $\phi(x)$ into an algebraic equation for $\mathcal{L}_\phi(p)$. We then need to perform an inverse Laplace transform to find $\phi(x)$.
For illustration, we will consider a simple example with $\tau(x) = 1$, $\sigma(x) = 1$, $\lambda = 0$. The differential equation then becomes

$$-\frac{d^2\phi(x)}{dx^2} = F(x), \tag{27}$$

where we will take the initial conditions to be $\phi(0) = 0$ and $d\phi(0)/dx = 0$. For our example, we will also take $F(x) = F_0 e^{-\gamma x}$. Multiplying, both sides of the equation by e^{-px} and integrating $0 \leq x \leq \infty$, we find

$$\mathcal{L}_\phi(p) = -\frac{F_0}{p^2(\gamma + p)}. \tag{28}$$

In general the inverse Laplace transform involves performing a contour integral, but we can use the following simple relations

$$\mathcal{L}_1 = \int_0^\infty e^{-px}dx = \frac{1}{p}. \tag{29}$$

$$\mathcal{L}_x = \int_0^\infty x e^{-px}dx = \frac{1}{p^2}. \tag{30}$$

$$\mathcal{L}_{e^{-\gamma x}} = \int_0^\infty e^{-\gamma x}e^{-px}dx = \frac{1}{p+\gamma}. \tag{31}$$

Noting that

$$-\frac{F_0}{p^2(\gamma+p)} = -\frac{F_0}{\gamma^2}\left(\frac{1}{\gamma+p} - \frac{1}{p} + \frac{\gamma}{p^2}\right), \tag{32}$$

we see that the inverse Laplace transform gives us

$$\phi(x) = \frac{F_0}{\gamma^2}\left(1 - e^{-\gamma x} - \gamma x\right). \tag{33}$$

We can check that this a solution to the differential equation

$$-\frac{d^2\phi}{dx^2} = F_0 e^{-\gamma x} \quad \text{for} \quad \phi(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(0) = 0$$

Using Laplace transforms to solve equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right) \quad \text{with} \quad \phi(0)=0,\ \frac{d\phi(0)}{dx}=0$$

$$\mathcal{L}_\phi(p)=-\left(\frac{\pi}{L}\right)\frac{F_0}{\left(p^2+1\right)\left(p^2+\left(\frac{\pi}{L}\right)^2\right)}$$

$$=-F_0\left(\frac{\pi/L}{(\pi/L)^2-1}\right)\left(\frac{1}{p^2+1}-\frac{1}{p^2+\left(\frac{\pi}{L}\right)^2}\right)$$

Note that : $\int_0^\infty \sin(at)e^{-pt}dt=\frac{a}{a^2+p^2}$

$$\Rightarrow \phi(x)=\frac{F_0}{(\pi/L)^2-1}\left(\sin\left(\frac{\pi x}{L}\right)-\frac{\pi}{L}\sin(x)\right)$$

Inverse Laplace transform :

$$\mathcal{L}_\phi(p)=\int_0^\infty e^{-pt}\phi(t)dt$$

$$\phi(t)=\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}\mathcal{L}_\phi(p)dp$$

Check : $\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}\mathcal{L}_\phi(p)dt=\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} e^{pt}dp\int_0^\infty e^{-pu}\phi(u)du$

$$\frac{1}{2\pi i}\int_0^\infty \phi(u)du\int_{\lambda-i\infty}^{\lambda+i\infty} e^{p(t-u)}dp=\frac{1}{2\pi i}\int_0^\infty \phi(u)du\int_{-\infty}^{\infty} e^{\lambda(t-u)}e^{is(t-u)}ids$$

$$=\frac{1}{2\pi i}\int_0^\infty \phi(u)du\left(e^{\lambda(t-u)}2\pi i\,\delta(t-u)\right)$$

$$=\begin{cases}\phi(t) & \text{if } t\geq 0\\ 0 & \text{otherwise}\end{cases}$$