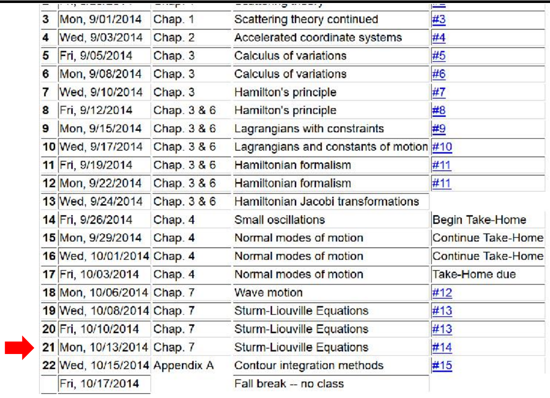
PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 21:

Summary of mathematical methods – continued:

1. **Green's function methods**
2. **Laplace transform methods**
3. **Motivation for contour integration**



3	Mon, 9/01/2014	Chap. 1	Scattering theory continued	#3
4	Wed, 9/03/2014	Chap. 2	Accelerated coordinate systems	#4
5	Fri, 9/05/2014	Chap. 3	Calculus of variations	#5
6	Mon, 9/08/2014	Chap. 3	Calculus of variations	#6
7	Wed, 9/10/2014	Chap. 3	Hamilton's principle	#7
8	Fri, 9/12/2014	Chap. 3 & 6	Hamilton's principle	#8
9	Mon, 9/15/2014	Chap. 3 & 6	Lagrangians with constraints	#9
10	Wed, 9/17/2014	Chap. 3 & 6	Lagrangians and constants of motion	#10
11	Fri, 9/19/2014	Chap. 3 & 6	Hamiltonian formalism	#11
12	Mon, 9/22/2014	Chap. 3 & 6	Hamiltonian formalism	#11
13	Wed, 9/24/2014	Chap. 3 & 6	Hamiltonian Jacobi transformations	
14	Fri, 9/26/2014	Chap. 4	Small oscillations	Begin Take-Home
15	Mon, 9/29/2014	Chap. 4	Normal modes of motion	Continue Take-Home
16	Wed, 10/01/2014	Chap. 4	Normal modes of motion	Continue Take-Home
17	Fri, 10/03/2014	Chap. 4	Normal modes of motion	Take-Home due
18	Mon, 10/06/2014	Chap. 7	Wave motion	#12
19	Wed, 10/08/2014	Chap. 7	Sturm-Liouville Equations	#13
20	Fri, 10/10/2014	Chap. 7	Sturm-Liouville Equations	#13
21	Mon, 10/13/2014	Chap. 7	Sturm-Liouville Equations	#14
22	Wed, 10/15/2014	Appendix A	Contour integration methods	#15
	Fri, 10/17/2014		Fall break -- no class	

Recap: Sturm-Liouville equation (assume all functions and constants are real):

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi_0(x)=0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi(x)=F(x)$

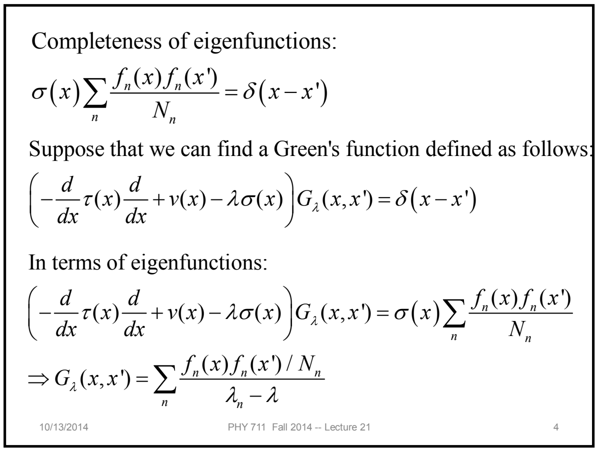
Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

We proved several properties of the eigenfunctions $f_n(x)$ – *including* orthogonality and completeness.

Orthogonality statement: $\int_a^b \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n,$

where $N_n \equiv \int_a^b \sigma(x)(f_n(x))^2\,dx.$



Completeness of eigenfunctions:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n} = \delta(x-x')$$

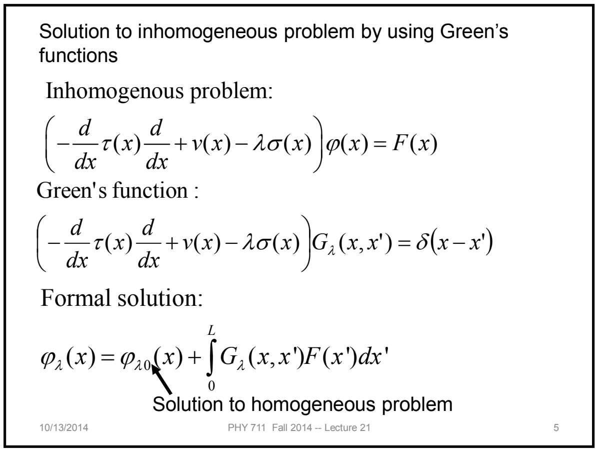
Suppose that we can find a Green's function defined as follows:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x') = \delta(x-x')$$

In terms of eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x') = \sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n}$$

$$\Rightarrow G_\lambda(x,x') = \sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}$$



Solution to inhomogeneous problem by using Green's functions

Inhomogenous problem:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x) = F(x)$$

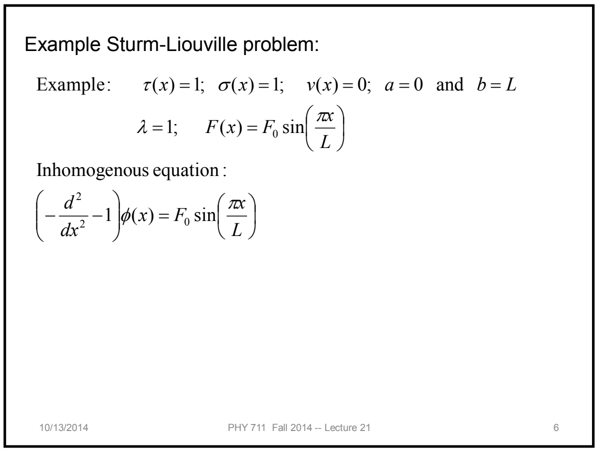
Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x') = \delta(x-x')$$

Formal solution:

$$\varphi_\lambda(x) = \varphi_{\lambda 0}(x) + \int_0^L G_\lambda(x,x')F(x')dx'$$

Solution to homogeneous problem



Example Sturm-Liouville problem:

Example: $\tau(x)=1;\ \sigma(x)=1;\ v(x)=0;\ a=0$ and $b=L$

$\lambda=1;\quad F(x)=F_0\sin\left(\frac{\pi x}{L}\right)$

Inhomogenous equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x) = F_0\sin\left(\frac{\pi x}{L}\right)$$

Eigenvalue equation :

$$\left(-\frac{d^2}{dx^2}\right)f_n(x)=\lambda_n f_n(x)$$

Eigenfunctions

$$f_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

Eigenvalues :

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2$$

Completeness of eigenfunctions :

$$\sigma(x)\sum_n\frac{f_n(x)f_n(x')}{N_n}=\delta(x-x')$$

In this example :

$$\frac{2}{L}\sum_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)=\delta(x-x')$$

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Green's function for the example :

$$G(x,x')=\sum_n\frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}=\frac{2}{L}\sum_n\frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}$$

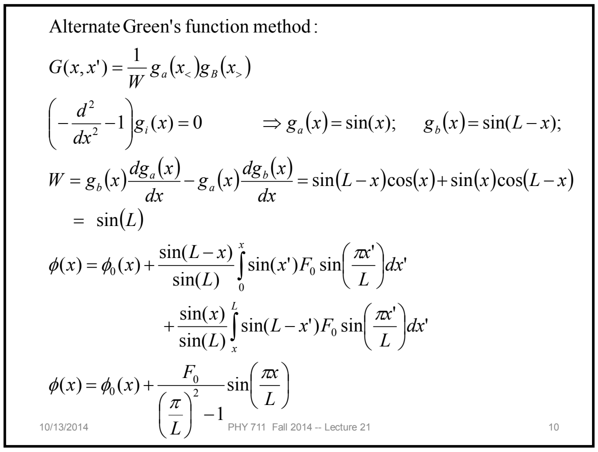
Using Green's function to solve inhomogenous equation :

$$\left(-\frac{d^2}{dx^2}-1\right)\phi(x)=F_0\sin\left(\frac{\pi x}{L}\right)$$

$$\phi(x)=\phi_0(x)+\int_0^L G(x,x')F_0\sin\left(\frac{\pi x'}{L}\right)dx'$$

$$=\phi_0(x)+\frac{2}{L}\sum_n\left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2-1}\int_0^L\sin\left(\frac{n\pi x'}{L}\right)F_0\sin\left(\frac{\pi x'}{L}\right)dx'\right]$$

$$=\phi_0(x)+\frac{F_0}{\left(\frac{\pi}{L}\right)^2-1}\sin\left(\frac{\pi x}{L}\right)$$



Alternate Green's function method:

$$G(x,x') = \frac{1}{W} g_a(x_<) g_B(x_>)$$

$$\left(-\frac{d^2}{dx^2} - 1\right) g_i(x) = 0 \qquad \Rightarrow g_a(x) = \sin(x); \quad g_b(x) = \sin(L-x);$$

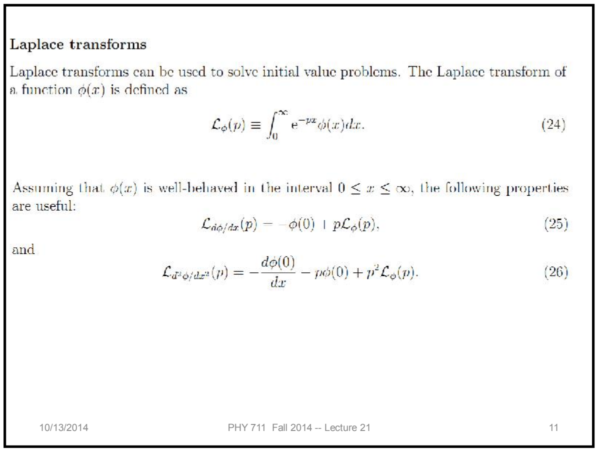
$$W = g_b(x)\frac{dg_a(x)}{dx} - g_a(x)\frac{dg_b(x)}{dx} = \sin(L-x)\cos(x) + \sin(x)\cos(L-x)$$

$$= \sin(L)$$

$$\phi(x) = \phi_0(x) + \frac{\sin(L-x)}{\sin(L)}\int_0^x \sin(x')F_0 \sin\left(\frac{\pi x'}{L}\right)dx'$$

$$+\frac{\sin(x)}{\sin(L)}\int_x^L \sin(L-x')F_0 \sin\left(\frac{\pi x'}{L}\right)dx'$$

$$\phi(x) = \phi_0(x) + \frac{F_0}{\left(\frac{\pi}{L}\right)^2 - 1}\sin\left(\frac{\pi x}{L}\right)$$



Laplace transforms

Laplace transforms can be used to solve initial value problems. The Laplace transform of a function $\phi(x)$ is defined as

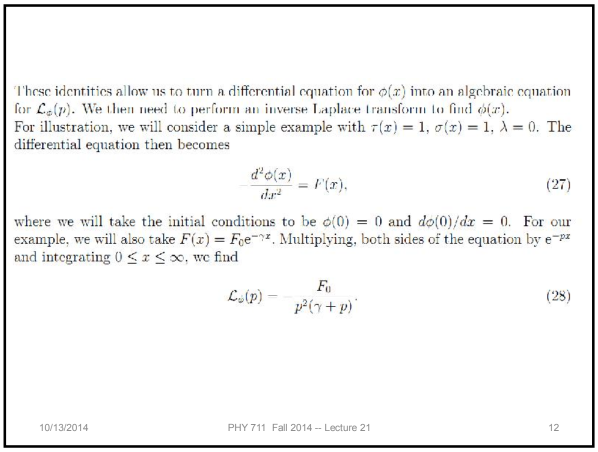
$$\mathcal{L}_\phi(p) \equiv \int_0^\infty e^{-px}\phi(x)dx. \tag{24}$$

Assuming that $\phi(x)$ is well-behaved in the interval $0 \le x \le \infty$, the following properties are useful:

$$\mathcal{L}_{d\phi/dx}(p) = -\phi(0) + p\mathcal{L}_\phi(p), \tag{25}$$

and

$$\mathcal{L}_{d^2\phi/dx^2}(p) = -\frac{d\phi(0)}{dx} - p\phi(0) + p^2\mathcal{L}_\phi(p). \tag{26}$$



These identities allow us to turn a differential equation for $\phi(x)$ into an algebraic equation for $\mathcal{L}_\phi(p)$. We then need to perform an inverse Laplace transform to find $\phi(x)$.
For illustration, we will consider a simple example with $\tau(x) = 1$, $\sigma(x) = 1$, $\lambda = 0$. The differential equation then becomes

$$-\frac{d^2\phi(x)}{dx^2} = F(x), \tag{27}$$

where we will take the initial conditions to be $\phi(0) = 0$ and $d\phi(0)/dx = 0$. For our example, we will also take $F(x) = F_0 e^{-\gamma x}$. Multiplying, both sides of the equation by e^{-px} and integrating $0 \le x \le \infty$, we find

$$\mathcal{L}_\phi(p) = -\frac{F_0}{p^2(\gamma + p)}. \tag{28}$$

In general the inverse Laplace transform involves performing a contour integral, but we can use the following simple relations

$$\mathcal{L}_1 = \int_0^\infty e^{-px} dx = \frac{1}{p}. \qquad (29)$$

$$\mathcal{L}_x = \int_0^\infty x e^{-px} dx = \frac{1}{p^2}. \qquad (30)$$

$$\mathcal{L}_{e^{-\gamma x}} = \int_0^\infty e^{-\gamma x} e^{-px} dx = \frac{1}{p+\gamma}. \qquad (31)$$

Noting that

$$-\frac{F_0}{p^2(\gamma+p)} = -\frac{F_0}{\gamma^2}\left(\frac{1}{\gamma+p} - \frac{1}{p} + \frac{\gamma}{p^2}\right), \qquad (32)$$

we see that the inverse Laplace transform gives us

$$\phi(x) = \frac{F_0}{\gamma^2}\left(1 - e^{-\gamma x} - \gamma x\right). \qquad (33)$$

We can check that this a solution to the differential equation

$$-\frac{d^2\phi}{dx^2} = F_0 e^{-\gamma x} \qquad \text{for} \quad \phi(0) = 0 \qquad \text{and} \quad \frac{d\phi}{dx}(0) = 0$$

Using Laplace transforms to solve equation :

$$\left(-\frac{d^2}{dx^2} - 1\right)\phi(x) = F_0 \sin\left(\frac{\pi x}{L}\right) \qquad \text{with} \quad \phi(0) = 0, \; \frac{d\phi(0)}{dx} = 0$$

$$\mathcal{L}_\phi(p) = -\left(\frac{\pi}{L}\right)\frac{F_0}{\left(p^2+1\right)\left(p^2 + \left(\frac{\pi}{L}\right)^2\right)}$$

$$= -F_0\left(\frac{\pi/L}{(\pi/L)^2 - 1}\right)\left(\frac{1}{p^2+1} - \frac{1}{p^2 + \left(\frac{\pi}{L}\right)^2}\right)$$

Note that : $\int_0^\infty \sin(at) e^{-pt} dt = \frac{a}{a^2 + p^2}$

$$\Rightarrow \phi(x) = \frac{F_0}{(\pi/L)^2 - 1}\left(\sin\left(\frac{\pi x}{L}\right) - \frac{\pi}{L}\sin(x)\right)$$

Inverse Laplace transform :

$$\mathcal{L}_\phi(p) = \int_0^\infty e^{-pt}\phi(t) dt$$

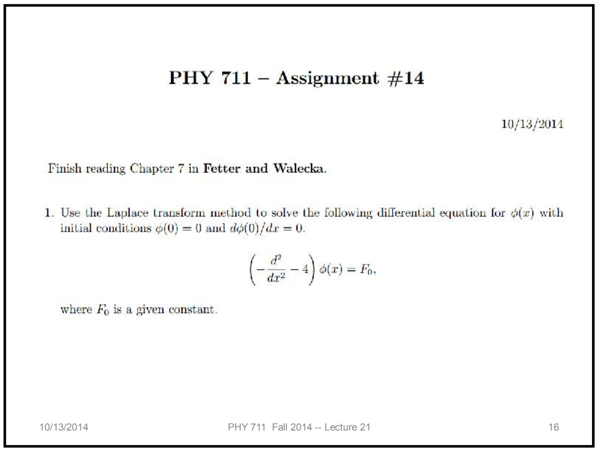
$$\phi(t) = \frac{1}{2\pi i}\int_{\lambda - i\infty}^{\lambda + i\infty} e^{pt}\mathcal{L}_\phi(p) dp$$

Check : $\frac{1}{2\pi i}\int_{\lambda - i\infty}^{\lambda + i\infty} e^{pt}\mathcal{L}_\phi(p) dt = \frac{1}{2\pi i}\int_{\lambda - i\infty}^{\lambda + i\infty} e^{pt} dp \int_0^\infty e^{-pu}\phi(u) du$

$$\frac{1}{2\pi i}\int_0^\infty \phi(u) du \int_{\lambda - i\infty}^{\lambda + i\infty} e^{p(t-u)} dp = \frac{1}{2\pi i}\int_0^\infty \phi(u) du \int_{-\infty}^{\infty} e^{\lambda(t-u)} e^{is(t-u)} i ds$$

$$= \frac{1}{2\pi i}\int_0^\infty \phi(u) du \left(e^{\lambda(t-u)} 2\pi i \, \delta(t-u)\right)$$

$$= \begin{cases} \phi(t) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



PHY 711 – Assignment #14

10/13/2014

Finish reading Chapter 7 in **Fetter and Walecka**.

1. Use the Laplace transform method to solve the following differential equation for $\phi(x)$ with initial conditions $\phi(0) = 0$ and $d\phi(0)/dx = 0$.

$$\left(-\frac{d^2}{dx^2} - 4\right)\phi(x) = F_0,$$

where F_0 is a given constant.