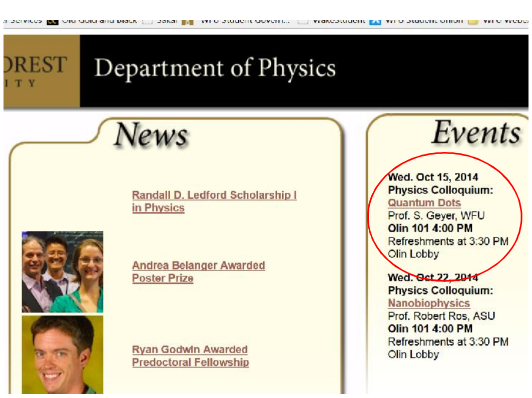
PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 22:

Summary of some mathematical tools

1. Contour integration
2. Fourier transforms
3. Fast Fourier transforms

3	Mon, 9/01/2014	Chap. 1	Scattering theory continued	#3
4	Wed, 9/03/2014	Chap. 2	Accelerated coordinate systems	#4
5	Fri, 9/05/2014	Chap. 3	Calculus of variations	#5
6	Mon, 9/08/2014	Chap. 3	Calculus of variations	#6
7	Wed, 9/10/2014	Chap. 3	Hamilton's principle	#7
8	Fri, 9/12/2014	Chap. 3 & 6	Hamilton's principle	#8
9	Mon, 9/15/2014	Chap. 3 & 6	Lagrangians with constraints	#9
10	Wed, 9/17/2014	Chap. 3 & 6	Lagrangians and constants of motion	#10
11	Fri, 9/19/2014	Chap. 3 & 6	Hamiltonian formalism	#11
12	Mon, 9/22/2014	Chap. 3 & 6	Hamiltonian formalism	#11
13	Wed, 9/24/2014	Chap. 3 & 6	Hamiltonian Jacobi transformations	
14	Fri, 9/26/2014	Chap. 4	Small oscillations	Begin Take-Home
15	Mon, 9/29/2014	Chap. 4	Normal modes of motion	Continue Take-Home
16	Wed, 10/01/2014	Chap. 4	Normal modes of motion	Continue Take-Home
17	Fri, 10/03/2014	Chap. 4	Normal modes of motion	Take-Home due
18	Mon, 10/06/2014	Chap. 7	Wave motion	#12
19	Wed, 10/08/2014	Chap. 7	Sturm-Liouville Equations	#13
20	Fri, 10/10/2014	Chap. 7	Sturm-Liouville Equations	#13
21	Mon, 10/13/2014	Chap. 7	Sturm-Liouville Equations	#14
22	Wed, 10/15/2014	Appendix A	Contour integration methods	#15
	Fri, 10/17/2014		Fall break -- no class	



PHY 711 – Contour Integration

These notes summarize some basic properties of complex functions and their integrals. An *analytic* function $f(z)$ in a certain region of the complex plane z is one which takes a single (non-infinite) value and is differentiable within that region. Cauchy's theorm states that a closed contour integral of the function within that region has the value

$$\oint_C f(z) = 0. \tag{1}$$

As an example, functions composed of integer powers of z –

$$f(z) = z^n, \quad \text{for} \quad n = 0, 1, +2, +3.... \tag{2}$$

fall in this category. Notice that non-integral powers are generally not analytic and that $n = -1$ is also special. In fact, we can show that

$$\oint_C \frac{dz}{z} = 2\pi i. \tag{3}$$

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This result follows from the fact that we can deform the contour to a unit circle about the origin so that $z = e^{i\theta}$. Then

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta}} i d\theta = 2\pi i. \tag{4}$$

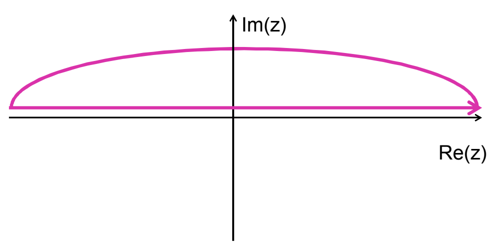
One result of this analysis is the Cauchy integral formula which states that for any analytic function $f(z)$ within a region C,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} dz'. \tag{5}$$

Example

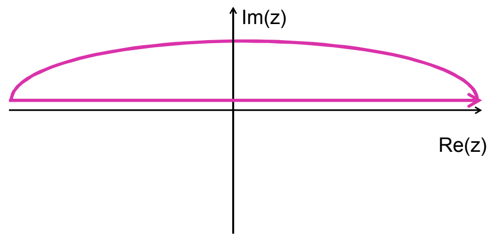
Suppose $f(|z| \to \infty) = 0$ and for $z = x$:

$$f(x) = a(x) + ib(x)$$



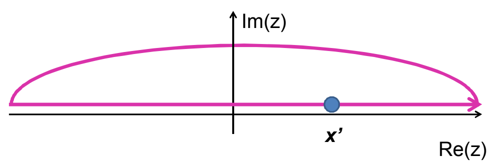
Example -- continued

$$f(z)=\frac{1}{2\pi i}\oint\frac{f(z')}{z'-z}dz' \quad \text{where} \quad f(x)=a(x)+ib(x)$$



$$a(x)+ib(x)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{a(x')+ib(x')}{x'-x}dx'$$

Example -- continued



$$\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx'=\int_{-\infty}^{x-\varepsilon}\frac{f(x')}{x'-x}dx'+\int_{x+\varepsilon}^{\infty}\frac{f(x')}{x'-x}dx'+\int_{x-\varepsilon}^{x+\varepsilon}\frac{f(x')}{x'-x}dx'$$

$$=P\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx'+i\pi\, f(x)$$

Example -- continued

$$\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx'=\int_{-\infty}^{x-\varepsilon}\frac{f(x')}{x'-x}dx'+\int_{x+\varepsilon}^{\infty}\frac{f(x')}{x'-x}dx'+\int_{x-\varepsilon}^{x+\varepsilon}\frac{f(x')}{x'-x}dx'$$

$$=P\int_{-\infty}^{\infty}\frac{f(x')}{x'-x}dx'+i\pi\, f(x)$$

$$a(x)+ib(x)=\frac{P}{2\pi i}\int_{-\infty}^{\infty}\frac{a(x')+ib(x')}{x'-x}dx'+\frac{\pi i}{2\pi i}\left(a(x)+ib(x)\right)$$

$$\Rightarrow a(x)=\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{b(x')}{x'-x}dx' \qquad b(x)=-\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{a(x')}{x'-x}dx'$$

Kramers-Kronig relationship

Another result of this analysis is the Residue Theorm which states that if the complex function $g(z)$ has poles at a finite number of points z_p within a region C but is otherwise analytic, the contour integral can be evaluated according to

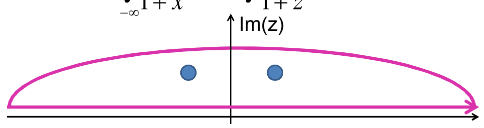
$$\oint_C g(z)dz = 2\pi i \sum_p Res(g_p), \tag{6}$$

where the residue is given by

$$Res(g_p) \equiv \lim_{z\to z_p}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left((z-z_p)^m g(z)\right)\right\}, \tag{7}$$

where m denotes the order of the pole.

Example: $\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}dx = \oint \frac{z^2}{1+z^4}dz$



$$\oint \frac{z^2}{1+z^4}dz = 2\pi i\left(\text{Res}\left(z_p = e^{i\pi/4}\right) + \text{Res}\left(z_p = e^{3i\pi/4}\right)\right)$$

$$1+z^4 = \left(z-e^{i\pi/4}\right)\left(z-e^{3i\pi/4}\right)\left(z-e^{-i\pi/4}\right)\left(z-e^{-3i\pi/4}\right)$$

$$\text{Res}\left(z_p = e^{i\pi/4}\right) = \frac{e^{i\pi/4}}{4i} \quad \text{Res}\left(z_p = e^{3i\pi/4}\right) = -\frac{e^{3i\pi/4}}{4i}$$

$$\oint \frac{z^2}{1+z^4}dz = 2\pi i\left(\frac{e^{i\pi/4}}{4i} - \frac{e^{3i\pi/4}}{4i}\right) = \frac{\pi}{\sqrt{2}}$$

Contour integral for homework:

$$\int_0^\infty \frac{\cos(ax)}{4x^4+5x^2+1}dx.$$

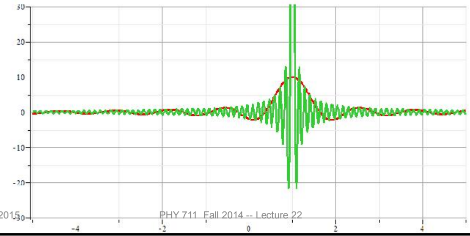
Fourier transforms

A useful identity

$$\int_{-\infty}^{\infty} dt\, e^{-i(\omega-\omega_0)t} = 2\pi\delta(\omega-\omega_0)$$

Note that

$$\int_{-T}^{T} dt\, e^{-i(\omega-\omega_0)t} = \frac{2\sin[(\omega-\omega_0)T]}{\omega-\omega_0}$$



Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega\, F(\omega)\, e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t)\, e^{i\omega t}$$

Check:

$$f(t) = \int_{-\infty}^{\infty} d\omega \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dt'\, f(t')\, e^{i\omega t'}\right) e^{-i\omega t}$$

$$f(t) = \int_{-\infty}^{\infty} dt'\, f(t') \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\; e^{i\omega(t'-t)}\right) = \int_{-\infty}^{\infty} dt'\, f(t')\, \delta(t'-t)$$

Note: The location of the 2π factor varies among texts.

Properties of Fourier transforms -- Parseval's theorem:

$$\int_{-\infty}^{\infty} dt\, (f(t))^* f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, (F(\omega))^* F(\omega)$$

Check:

$$\int_{-\infty}^{\infty} dt\, (f(t))^* f(t) = \int_{-\infty}^{\infty} dt \left(\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F(\omega) e^{i\omega t}\right)^* \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' F(\omega') e^{i\omega' t}\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F^*(\omega) \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' F(\omega') \int_{-\infty}^{\infty} dt\, e^{i(\omega'-\omega)t}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F^*(\omega) \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' F(\omega')\, 2\pi\delta(\omega'-\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F^*(\omega) F(\omega)$$

Use of Fourier transforms to solve wave equation

Wave equation : $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$

Suppose $u(x,t) = e^{-i\omega t}\tilde{F}(x,\omega)$ where $\tilde{F}(x,\omega)$ satisfies the equation :

$$\frac{\partial^2 \tilde{F}(x,\omega)}{\partial x^2} = -\frac{\omega^2}{c^2}\tilde{F}(x,\omega) \equiv -k^2\tilde{F}(x,\omega)$$

Further assume that fixed boundary conditions apply : $0 \le x \le L$

with $\tilde{F}(0,\omega) = 0$ and $\tilde{F}(L,\omega) = 0$

For $n = 1,2,3\cdots$

$$\tilde{F}_n(x,\omega) = \sin\left(\frac{n\pi x}{L}\right) \qquad k \to k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$$

$$u(x,t) = e^{-i\omega_n t}\sin(k_n x) = e^{-i\omega_n t}\frac{\left(e^{ik_n x} - e^{-ik_n x}\right)}{2i} = \frac{\left(e^{ik_n(x-ct)} - e^{-ik_n(x+ct)}\right)}{2i}$$

Use of Fourier transforms to solve wave equation -- continued

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Using superposition : Suppose $u(x,t) = \sum_n C_n e^{-i\omega_n t}\tilde{F}_n(x,\omega_n)$

$$\frac{\partial^2 \tilde{F}_n(x,\omega_n)}{\partial x^2} = -\frac{\omega_n^2}{c^2}\tilde{F}(x,\omega_n) \equiv -k_n^2\tilde{F}(x,\omega_n)$$

For $\tilde{F}_n(x,\omega) = \sin\left(\frac{n\pi x}{L}\right) \qquad k \to k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$

$$\Rightarrow u(x,t) = \sum_n C_n e^{-i\omega_n t}\sin(k_n x) = \sum_n \frac{C_n}{2i}e^{-i\omega_n t}\left(e^{ik_n x} - e^{-ik_n x}\right)$$

$$= \sum_n \frac{C_n}{2i}\left(e^{ik_n(x-ct)} - e^{-ik_n(x+ct)}\right) \equiv f(x-ct) + g(x+ct)$$

Fourier transform for periodic function :

Suppose $f(t+nT) = f(t)$ for and integer n

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t)e^{i\omega t} = \sum_{n=-\infty}^{\infty}\left(\int_0^T dt\, f(t)e^{i\omega(t+nT)}\right)$$

Note that :

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega\sum_{\nu=-\infty}^{\infty}\delta(\omega - \nu\Omega), \quad \text{where } \Omega \equiv \frac{2\pi}{T}$$

Some details :

$$\sum_{n=-M}^{M} e^{in\omega T} = \frac{\sin\left(\left(M+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}$$

$$\lim_{M\to\infty}\left(\frac{\sin\left(\left(M+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}\right) = 2\pi\sum_{\nu}\delta(\omega T-\nu\Omega T) = \frac{2\pi}{T}\sum_{\nu}\delta(\omega-\nu\Omega)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega\sum_{\nu=-\infty}^{\infty}\delta(\omega-\nu\Omega), \quad \text{where} \quad \Omega \equiv \frac{2\pi}{T}$$

$$\Rightarrow F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t} = \sum_{\nu=-\infty}^{\infty}\Omega\delta(\omega-\nu\Omega)\left(\int_0^T dt\, f(t)e^{i\omega t}\right)$$

Thus, for a periodic function

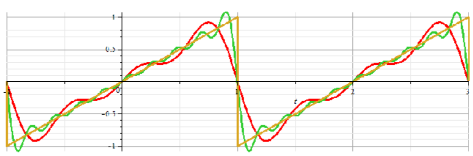
$$f(t) = \sum_{\nu=-\infty}^{\infty} F(\nu\Omega) e^{-i\nu\Omega t}$$

Example:

Suppose : $f(t) = \dfrac{t-nT}{T}$ for $(n-1)T \le t \le (n+1)T$; $n = 0,2,4,6\ldots$

Note, in this case the repeat period is $2T$ and the convenient sample time interval is $-T \le t \le T$.

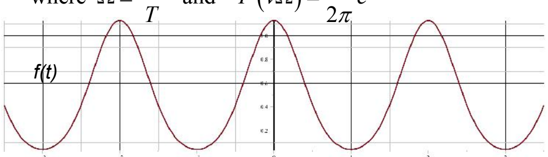
$$\bar{F}(\nu\Omega) = \frac{2\pi}{2T} i \int_{-T}^{T} \frac{t}{T}\sin\left(\frac{\nu\pi t}{2T}\right)dt \qquad f(t) = \sum_{\nu=1}^{\infty} 2\left|\bar{F}(\nu\Omega)\right|\sin\left(\frac{\nu\pi t}{2T}\right)$$

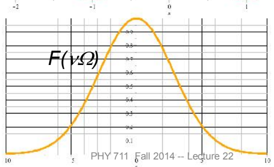


Example:

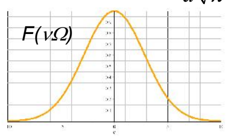
Suppose: $f(t) = \dfrac{1}{a\sqrt{\pi}}\displaystyle\sum_{n=-\infty}^{\infty} e^{-(t+nT)^2/a^2} = \sum_{\nu=-\infty}^{\infty} F(\nu\Omega) e^{-i\nu\Omega t}$

where $\Omega \equiv \dfrac{2\pi}{T}$ and $F(\nu\Omega) = \dfrac{1}{2\pi} e^{-a^2\nu^2\Omega^2/4}$

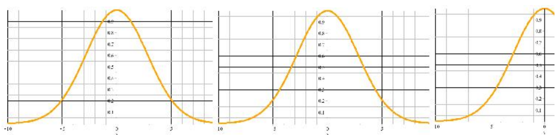




Continued: $f(t)=\frac{1}{a\sqrt{\pi}}\sum_{n=-\infty}^{\infty} e^{-(t+nT)^2/a^2} = \sum_{\nu=-\infty}^{\infty} F(\nu\Omega)e^{-i\nu\Omega t}$



Note: $f(t)\approx=\sum_{\nu=-M}^{M} F(\nu\Omega)e^{-i\nu\Omega t}$



$$\Rightarrow f\left(\frac{mT}{2M+1}\right)=\sum_{\nu=-M}^{M} F(\nu\Omega)e^{-i2\pi\nu m/(2M+1)}$$

Thus, for a periodic function

$$f(t)=\sum_{\nu=-\infty}^{\infty} F(\nu\Omega)e^{-i\nu\Omega t}$$

Now suppose that the transformed function is bounded;

$$|F(\nu\Omega)|\leq\varepsilon \quad \text{for} \quad |\nu|\geq N$$

Define a periodic transform function function

$$\tilde{F}(\nu\Omega)\equiv\tilde{F}(\nu\Omega+\nu'((2N+1)\Omega))$$

Effect on time domain :

$$f(t)=\sum_{\nu=-\infty}^{\infty}\tilde{F}(\nu\Omega)e^{-i\nu\Omega t} = \frac{2\pi}{(2N+1)\Omega}\sum_{\nu=-N}^{N}\tilde{F}(\nu\Omega)e^{-i\nu\Omega t}\sum_{\mu}\delta\left(t-\frac{\mu T}{2N+1}\right)$$

Doubly periodic functions

$$t\rightarrow\frac{\mu T}{2N+1}$$

$$\tilde{f}_\mu=\frac{1}{2N+1}\sum_{\nu=-N}^{N}\tilde{F}_\nu e^{-i2\pi\nu\mu/(2N+1)}$$

$$\tilde{F}_\nu=\sum_{\mu=-N}^{N}\tilde{f}_\mu e^{i2\pi\nu\mu/(2N+1)}$$

More convenient notation

$2N+1 \to M$

$$\tilde{f}_\mu = \frac{1}{M}\sum_{\nu=0}^{M-1}\tilde{F}_\nu e^{-i2\pi\nu\mu/M}$$

$$\tilde{F}_\nu = \sum_{\mu=0}^{M}\tilde{f}_\mu e^{i2\pi\nu\mu/M}$$

Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \cdots$$

$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \cdots$$

$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \cdots$$

Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \cdots$$

$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \cdots$$

$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \cdots$$

However, $W^M = \left(e^{i2\pi/M}\right)^M = 1$

and $W^{M/2} = \left(e^{i2\pi/M}\right)^{M/2} = -1$

Cooley-Tukey algorithm: J. W. Cooley and J. W. Tukey, “An algorithm for machine calculation of complex Fourier series” Math. Computation 19, 297-301 (1965)