

**PHY 711 Classical Mechanics and
Mathematical Methods**
10-10:50 AM MWF Olin 103

Plan for Lecture 23:

More mathematical tools

1. Fourier transforms

2. Fast Fourier transforms

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4	Wed, 9/03/2014	Chap. 2	Accelerated coordinate systems	#4
5	Fri, 9/05/2014	Chap. 3	Calculus of variations	#5
6	Mon, 9/08/2014	Chap. 3	Calculus of variations	#6
7	Wed, 9/10/2014	Chap. 3	Hamilton's principle	#7
8	Fri, 9/12/2014	Chap. 3 & 6	Hamilton's principle	#8
9	Mon, 9/15/2014	Chap. 3 & 6	Lagrangians with constraints	#9
10	Wed, 9/17/2014	Chap. 3 & 6	Lagrangians and constants of motion	#10
11	Fri, 9/19/2014	Chap. 3 & 6	Hamiltonian formalism	#11
12	Mon, 9/22/2014	Chap. 3 & 6	Hamiltonian formalism	#11
13	Wed, 9/24/2014	Chap. 3 & 6	Hamiltonian Jacobi transformations	
14	Fri, 9/26/2014	Chap. 4	Small oscillations	Begin Take-Home
15	Mon, 9/29/2014	Chap. 4	Normal modes of motion	Continue Take-Home
16	Wed, 10/01/2014	Chap. 4	Normal modes of motion	Continue Take-Home
17	Fri, 10/03/2014	Chap. 4	Normal modes of motion	Take-Home due
18	Mon, 10/06/2014	Chap. 7	Wave motion	#12
19	Wed, 10/08/2014	Chap. 7	Sturm-Liouville Equations	#13
20	Fri, 10/10/2014	Chap. 7	Sturm-Liouville Equations	#13
21	Mon, 10/13/2014	Chap. 7	Sturm-Liouville Equations	#14
22	Wed, 10/15/2014	Appendix A	Contour integration methods	#15
23	Fri, 10/17/2014		Fall break -- no class	
23	Mon, 10/20/2014	Appendix A	Fourier transforms	#16

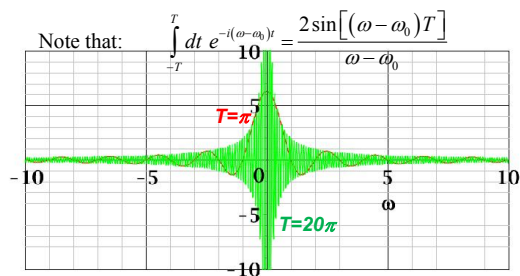
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Fourier transforms

A useful identity: $\int_{-\infty}^{\infty} dt e^{-i(\omega-\omega_0)t} = 2\pi\delta(\omega-\omega_0)$



For $T > 0$: $\int_{-\infty}^{\infty} \frac{2\sin((\omega-\omega_0)T)}{\omega-\omega_0} d\omega = 2\pi$

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Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega F(\omega) e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{i\omega t}$$

Check:

$$f(t) = \int_{-\infty}^{\infty} d\omega \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} dt' f(t') e^{i\omega t'} \right) e^{-i\omega t}$$

$$f(t) = \int_{-\infty}^{\infty} dt' f(t') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t'-t)} \right) = \int_{-\infty}^{\infty} dt' f(t') \delta(t'-t)$$

Note: The location of the 2π factor varies among texts.

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Properties of Fourier transforms -- Parseval's theorem:

$$\int_{-\infty}^{\infty} dt (f(t))^* f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega (F(\omega))^* F(\omega)$$

$$\begin{aligned} \text{Check: } \int_{-\infty}^{\infty} dt (f(t))^* f(t) &= \int_{-\infty}^{\infty} dt \left(\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' F(\omega') e^{i\omega' t} \right)^* \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega F(\omega) e^{-i\omega t} \right) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' F^*(\omega') \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega F(\omega) \int_{-\infty}^{\infty} dt e^{i(\omega'-\omega)t} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' F^*(\omega') \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega F(\omega) 2\pi \delta(\omega'-\omega) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega F^*(\omega) F(\omega) \end{aligned}$$

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Use of Fourier transforms to solve wave equation

$$\text{Wave equation: } \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Suppose $u(x, t) = e^{-i\omega t} \tilde{F}(x, \omega)$ where $\tilde{F}(x, \omega)$ satisfies the equation:

$$\frac{\partial^2 \tilde{F}(x, \omega)}{\partial x^2} = -\frac{\omega^2}{c^2} \tilde{F}(x, \omega) = -k^2 \tilde{F}(x, \omega)$$

Further assume that fixed boundary conditions apply: $0 \leq x \leq L$

with $\tilde{F}(0, \omega) = 0$ and $\tilde{F}(L, \omega) = 0$

For $n = 1, 2, 3, \dots$

$$\tilde{F}_n(x, \omega) = \sin\left(\frac{n\pi x}{L}\right) \quad k \rightarrow k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$$

$$u(x, t) = e^{-i\omega_n t} \sin(k_n x) = e^{-i\omega_n t} \frac{(e^{ik_n x} - e^{-ik_n x})}{2i} = \frac{(e^{ik_n(x-ct)} - e^{-ik_n(x+ct)})}{2i}$$

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Use of Fourier transforms to solve wave equation -- continued

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Using superposition: Suppose $u(x, t) = \sum_n C_n e^{-i\omega_n t} \tilde{F}_n(x, \omega_n)$

$$\frac{\partial^2 \tilde{F}_n(x, \omega_n)}{\partial x^2} = -\frac{\omega_n^2}{c^2} \tilde{F}_n(x, \omega_n) \equiv -k_n^2 \tilde{F}_n(x, \omega_n)$$

$$\text{For } \tilde{F}_n(x, \omega) = \sin\left(\frac{n\pi x}{L}\right) \quad k \rightarrow k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$$

$$\begin{aligned} \Rightarrow u(x, t) &= \sum_n C_n e^{-i\omega_n t} \sin(k_n x) = \sum_n \frac{C_n}{2i} e^{-i\omega_n t} (e^{ik_n x} - e^{-ik_n x}) \\ &= \sum_n \frac{C_n}{2i} (e^{ik_n(x-ct)} - e^{-ik_n(x+ct)}) \equiv f(x-ct) + g(x+ct) \end{aligned}$$

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Fourier transform for periodic function:

Suppose $f(t + nT) = f(t)$ for and integer n

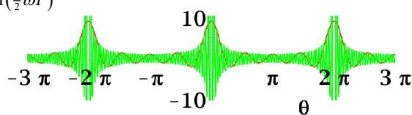
$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{i\omega t} = \sum_n \left(\int_0^T dt f(t) e^{i\omega(t+nT)} \right)$$

Note that:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega - \nu\Omega), \quad \text{where } \Omega \equiv \frac{2\pi}{T}$$

Details:

$$\sum_{n=-N}^N e^{in\omega T} = \frac{\sin\left((N + \frac{1}{2})\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}$$



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Some details:

$$\sum_{n=-M}^M e^{in\omega T} = \frac{\sin\left((M + \frac{1}{2})\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}$$

$$\lim_{M \rightarrow \infty} \left(\frac{\sin\left((M + \frac{1}{2})\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)} \right) = 2\pi \sum_{\nu} \delta(\omega T - \nu\Omega T) = \frac{2\pi}{T} \sum_{\nu} \delta(\omega - \nu\Omega)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega - \nu\Omega), \quad \text{where } \Omega \equiv \frac{2\pi}{T}$$

$$\Rightarrow F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{i\omega t} = \sum_{\nu=-\infty}^{\infty} \Omega \delta(\omega - \nu\Omega) \left(\int_0^T dt f(t) e^{i\omega t} \right)$$

Thus, for a periodic function

$$f(t) = \sum_{\nu=-\infty}^{\infty} F(\nu\Omega) e^{-i\nu\Omega t}$$

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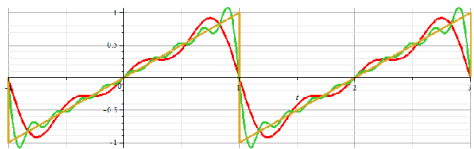
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Example:

Suppose: $f(t) = \frac{t-nT}{T}$ for $(n-1)T \leq t \leq (n+1)T$; $n = 0, 2, 4, 6 \dots$

Note, in this case the repeat period is $2T$ and the convenient sample time interval is $-T \leq t \leq T$.

$$\bar{F}(\nu\Omega) = \frac{2\pi}{2T} i \int_{-T}^T \frac{t}{T} \sin\left(\frac{\nu\pi t}{2T}\right) dt \quad f(t) = \sum_{\nu=1}^{\infty} 2|\bar{F}(\nu\Omega)| \sin\left(\frac{\nu\pi t}{2T}\right)$$



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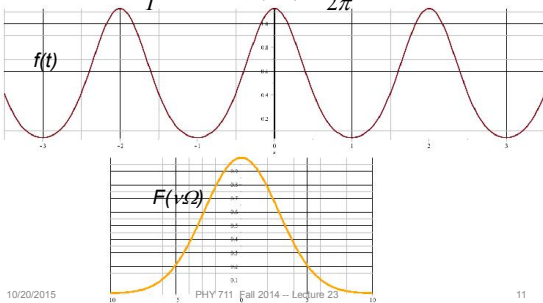
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Example:

$$\text{Suppose: } f(t) = \frac{1}{a\sqrt{\pi}} \sum_{n=-\infty}^{\infty} e^{-(t+nT)^2/a^2} = \sum_{\nu=-\infty}^{\infty} F(\nu\Omega) e^{-i\nu\Omega t}$$

$$\text{where } \Omega \equiv \frac{2\pi}{T} \text{ and } F(\nu\Omega) = \frac{1}{2\pi} e^{-a^2 \nu^2 \Omega^2 / 4}$$

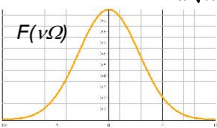


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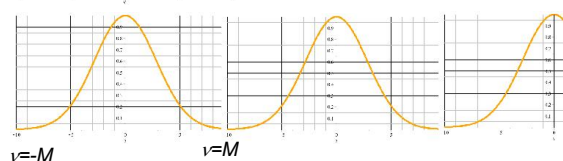
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$$\text{Continued: } f(t) = \frac{1}{a\sqrt{\pi}} \sum_{n=-\infty}^{\infty} e^{-(t+nT)^2/a^2} = \sum_{\nu=-\infty}^{\infty} F(\nu\Omega) e^{-i\nu\Omega t}$$



$$\text{Note: } f(t) \approx \sum_{\nu=-M}^M F(\nu\Omega) e^{-i\nu\Omega t}$$

 $\nu=-M$ $\nu=M$

$$\Rightarrow f\left(\frac{mT}{2M+1}\right) = \sum_{\nu=-M}^M F(\nu\Omega) e^{-i2\pi\nu m/(2M+1)}$$

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Thus, for a periodic function

$$f(t) = \sum_{\nu=-\infty}^{\infty} F(\nu\Omega) e^{-i\nu\Omega t}$$

Now suppose that the transformed function is bounded;

$$|F(\nu\Omega)| \leq \varepsilon \quad \text{for } |\nu| \geq N$$

Define a periodic transform function

$$\tilde{F}(\nu\Omega) \equiv \tilde{F}(\nu\Omega + \nu'((2N+1)\Omega))$$

Effect on time domain :

$$f(t) = \sum_{\nu=-\infty}^{\infty} \tilde{F}(\nu\Omega) e^{-i\nu\Omega t} = \frac{2\pi}{(2N+1)\Omega} \sum_{\nu=-N}^N \tilde{F}(\nu\Omega) e^{-i\nu\Omega t} \sum_{\mu} \delta\left(t - \frac{\mu T}{2N+1}\right)$$

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Doubly periodic functions

$$t \rightarrow \frac{\mu T}{2N+1}$$

$$\tilde{f}_{\mu} = \frac{1}{2N+1} \sum_{\nu=-N}^N \tilde{F}_{\nu} e^{-i2\pi\nu\mu/(2N+1)}$$

$$\tilde{F}_{\nu} = \sum_{\mu=-N}^N \tilde{f}_{\mu} e^{i2\pi\nu\mu/(2N+1)}$$

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More convenient notation

$$2N+1 \rightarrow M$$

$$\tilde{f}_{\mu} = \frac{1}{M} \sum_{\nu=0}^{M-1} \tilde{F}_{\nu} e^{-i2\pi\nu\mu/M}$$

$$\tilde{F}_{\nu} = \sum_{\mu=0}^{M-1} \tilde{f}_{\mu} e^{i2\pi\nu\mu/M}$$

Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \dots$$

$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \dots$$

$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \dots$$

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Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \dots$$

$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \dots$$

$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \dots$$

However, $W^M = (e^{i2\pi/M})^M = 1$

and $W^{M/2} = (e^{i2\pi/M})^{M/2} = -1$

Cooley-Tukey algorithm: J. W. Cooley and J. W. Tukey, "An algorithm for machine calculation of complex Fourier series" Math. Computation 19, 297-301 (1965)

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Example for $M = 4$

$$W^k = e^{-i2\pi k/4}$$

$$\begin{pmatrix} \tilde{F}_0 \\ \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \end{pmatrix} = \begin{pmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & W^1 & W^2 & W^3 \\ W^0 & W^2 & W^4 & W^6 \\ W^0 & W^3 & W^6 & W^9 \end{pmatrix} \begin{pmatrix} \tilde{f}_0 \\ \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{pmatrix}$$

16 complex multiplications and additions

Some simplifications:

$$\begin{pmatrix} \tilde{F}_0 \\ \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & W^1 & W^2 & W^3 \\ 1 & W^2 & 1 & W^2 \\ 1 & W^3 & W^2 & W^1 \end{pmatrix} \begin{pmatrix} \tilde{f}_0 \\ \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{pmatrix}$$

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Example for $M = 4$ continued

$$\begin{pmatrix} \tilde{F}_0 \\ \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & W^1 & W^2 & W^3 \\ 1 & W^2 & 1 & W^2 \\ 1 & W^3 & W^2 & W^1 \end{pmatrix} \begin{pmatrix} \tilde{f}_0 \\ \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{pmatrix}$$

Reorder and factor:

$$\begin{pmatrix} \tilde{F}_0 \\ \tilde{F}_2 \\ \tilde{F}_1 \\ \tilde{F}_3 \end{pmatrix} = \begin{pmatrix} 1 & W^0 & 0 & 0 \\ 1 & W^2 & 0 & 0 \\ 0 & 0 & 1 & W^1 \\ 0 & 0 & 1 & W^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & W^0 & 0 \\ 0 & 1 & 0 & W^0 \\ 1 & 0 & W^2 & 0 \\ 0 & 1 & 0 & W^2 \end{pmatrix} \begin{pmatrix} \tilde{f}_0 \\ \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{pmatrix}$$

4 complex multiplications and 8 complex additions

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