

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF Olin 103**

Plan for Lecture 25:

Rotational motion (Chapter 5)

**1. Rigid body motion in body fixed
frame**

**2. Conversion between body and
inertial reference frames**

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7	Wed, 9/10/2014	Chap. 3	Hamilton's principle	#7
8	Fri, 9/12/2014	Chap. 3 & 6	Hamilton's principle	#8
9	Mon, 9/15/2014	Chap. 3 & 6	Lagrangians with constraints	#9
10	Wed, 9/17/2014	Chap. 3 & 6	Lagrangians and constants of motion	#10
11	Fri, 9/19/2014	Chap. 3 & 6	Hamiltonian formalism	#11
12	Mon, 9/22/2014	Chap. 3 & 6	Hamiltonian formalism	#11
13	Wed, 9/24/2014	Chap. 3 & 6	Hamiltonian Jacobi transformations	
14	Fri, 9/26/2014	Chap. 4	Small oscillations	Begin Take-Home
15	Mon, 9/29/2014	Chap. 4	Normal modes of motion	Continue Take-Home
16	Wed, 10/01/2014	Chap. 4	Normal modes of motion	Continue Take-Home
17	Fri, 10/03/2014	Chap. 4	Normal modes of motion	Take-Home due
18	Mon, 10/06/2014	Chap. 7	Wave motion	#12
19	Wed, 10/08/2014	Chap. 7	Sturm-Liouville Equations	#13
20	Fri, 10/10/2014	Chap. 7	Sturm-Liouville Equations	#13
21	Mon, 10/13/2014	Chap. 7	Sturm-Liouville Equations	#14
22	Wed, 10/15/2014	Appendix A	Contour integration methods	#15
	Fri, 10/17/2014		Fall break - no class	
23	Mon, 10/20/2014	Appendix A	Fourier transforms	#16
24	Wed, 10/22/2014	Chap. 5	Motion of Rigid Bodies	#17
25	Fri, 10/24/2014	Chap. 5	Motion of Rigid Bodies	#18
26	Mon, 10/27/2014			

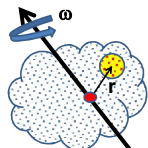
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Summary of previous results
describing rigid bodies rotating
about a fixed origin •

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{inertial}} = \boldsymbol{\omega} \times \mathbf{r}$$



$$\text{Kinetic energy: } T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\left| \boldsymbol{\omega} \times \mathbf{r}_p \right| \right)^2$$

$$= \sum_p \frac{1}{2} m_p (\boldsymbol{\omega} \times \mathbf{r}_p) \cdot (\boldsymbol{\omega} \times \mathbf{r}_p)$$

$$= \sum_p \frac{1}{2} m_p \left[(\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\mathbf{r}_p \cdot \mathbf{r}_p) - (\mathbf{r}_p \cdot \boldsymbol{\omega})^2 \right]$$

$$= \boldsymbol{\omega} \cdot \tilde{\mathbf{I}} \cdot \boldsymbol{\omega} \quad \tilde{\mathbf{I}} \equiv \sum_p m_p (\mathbf{r}_p^2 \mathbf{1} - \mathbf{r}_p \mathbf{r}_p)$$

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Moment of inertia tensor
Matrix notation:

$$\tilde{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \quad I_{ij} \equiv \sum_p m_p (\delta_{ij} r_p^2 - r_{pi} r_{pj})$$

For general coordinate system: $T = \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j$

For (body fixed) coordinate system that diagonalizes
moment of inertia tensor: $\tilde{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \quad i = 1, 2, 3$

$$\boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3 \quad \Rightarrow T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2$$

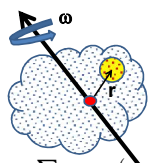
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Summary of previous results
describing rigid bodies rotating
about a fixed origin

$$\left(\frac{d\mathbf{r}}{dt} \right)_{\text{inertial}} = \boldsymbol{\omega} \times \mathbf{r}$$



$$\begin{aligned} \text{Angular momentum: } \mathbf{L} &= \sum_p m_p \mathbf{r}_p \times \mathbf{v}_p = \sum_p m_p \mathbf{r}_p \times (\boldsymbol{\omega} \times \mathbf{r}_p) \\ &= \sum_p m_p [\boldsymbol{\omega} (\mathbf{r}_p \cdot \mathbf{r}_p) - \mathbf{r}_p (\mathbf{r}_p \cdot \boldsymbol{\omega})] \\ &= \tilde{\mathbf{I}} \cdot \boldsymbol{\omega} \quad \tilde{\mathbf{I}} \equiv \sum_p m_p (1 r_p^2 - \mathbf{r}_p \mathbf{r}_p) \end{aligned}$$

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Descriptions of rotation about a given origin -- continued

For (body fixed) coordinate system that diagonalizes
moment of inertia tensor:

$$\tilde{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \quad \boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\text{Time derivative: } \frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt} \right)_{\text{body}} + \boldsymbol{\omega} \times \mathbf{L}$$

$$\frac{d\mathbf{L}}{dt} = I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 +$$

$$\tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) \hat{\mathbf{e}}_1 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) \hat{\mathbf{e}}_3$$

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Descriptions of rotation about a given origin -- continued
Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt} \right)_{\text{body}} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

For $\boldsymbol{\tau} = 0$ we can solve the Euler equations:

$$\begin{aligned} \frac{d\mathbf{L}}{dt} = 0 &= I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 + \\ &\quad \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) \hat{\mathbf{e}}_1 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) \hat{\mathbf{e}}_3 \\ I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) &= 0 \\ I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) &= 0 \\ I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) &= 0 \end{aligned}$$

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Euler equations for rotation in body fixed frame :

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

Solution for symmetric top -- $I_2 = I_1$:

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_1) = 0$$

$$I_1 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 = 0 \quad \Rightarrow \quad \tilde{\omega}_3 = (\text{constant})$$

$$\text{Define : } \Omega \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_1} \quad \begin{aligned} \dot{\tilde{\omega}}_1 &= -\tilde{\omega}_2 \Omega \\ \dot{\tilde{\omega}}_2 &= \tilde{\omega}_1 \Omega \end{aligned}$$

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Solution of Euler equations for a symmetric top -- continued

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2 \Omega \quad \dot{\tilde{\omega}}_2 = \tilde{\omega}_1 \Omega$$

$$\text{where } \Omega \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_1}$$

$$\text{Solution : } \tilde{\omega}_1(t) = A \cos(\Omega t + \varphi)$$

$$\tilde{\omega}_2(t) = A \sin(\Omega t + \varphi)$$

$$T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2 = \frac{1}{2} I_1 A^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2$$

$$\begin{aligned} \mathbf{L} &= I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3 \\ &= I_1 A (\cos(\Omega t + \varphi) \hat{\mathbf{e}}_1 + \sin(\Omega t + \varphi) \hat{\mathbf{e}}_2) + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3 \end{aligned}$$

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Euler equations for rotation in body fixed frame :

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

Solution for asymmetric top -- $I_3 \neq I_2 \neq I_1$:

Suppose : $\dot{\tilde{\omega}}_3 \approx 0$ Define : $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$

Define : $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

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Euler equations for asymmetric top -- continued

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

If $\dot{\tilde{\omega}}_3 \approx 0$, Define: $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$ $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

$$\dot{\tilde{\omega}}_1 = -\Omega_1 \tilde{\omega}_2 \quad \dot{\tilde{\omega}}_2 = \Omega_2 \tilde{\omega}_1$$

If Ω_1 and Ω_2 are both positive or both negative :

$$\tilde{\omega}_1(t) \approx A \cos(\sqrt{\Omega_1 \Omega_2} t + \varphi)$$

$$\tilde{\omega}_2(t) \approx A \sqrt{\frac{\Omega_2}{\Omega_1}} \sin(\sqrt{\Omega_1 \Omega_2} t + \varphi)$$

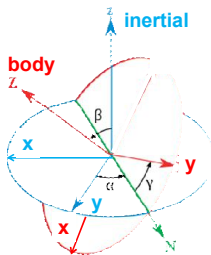
$\Rightarrow$ If Ω_1 and Ω_2 have opposite signs, solution is unstable.

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Transformation between body-fixed and inertial coordinate systems -- Euler angles

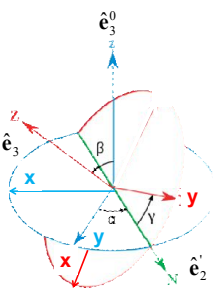


http://en.wikipedia.org/wiki/Euler_angles

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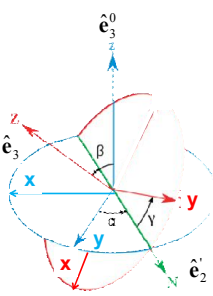
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$$\tilde{\omega} = \dot{\alpha} \hat{e}_3^0 + \dot{\beta} \hat{e}_2' + \dot{\gamma} \hat{e}_3$$

Need to express all components in body-fixed frame:

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$$\tilde{\omega} = \dot{\alpha} \hat{e}_3^0 + \dot{\beta} \hat{e}_2' + \dot{\gamma} \hat{e}_3$$

$$\hat{e}_2' = \sin \gamma \hat{e}_1 + \cos \gamma \hat{e}_2$$

Matrix representation :

$$\hat{e}_2' = \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin \gamma \\ \cos \gamma \\ 0 \end{pmatrix}$$

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$$\tilde{\omega} = \dot{\alpha} \hat{e}_3^0 + \dot{\beta} \hat{e}_2' + \dot{\gamma} \hat{e}_3$$

$$\hat{e}_3^0 = -\sin \beta \hat{e}_1' + \cos \beta \hat{e}_3' = -\cos \gamma \sin \beta \hat{e}_1 + \sin \gamma \sin \beta \hat{e}_2 + \cos \beta \hat{e}_3$$

Matrix representation:

$$\hat{e}_3^0 = \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin \beta \cos \gamma \\ \sin \beta \sin \gamma \\ \cos \beta \end{pmatrix}$$

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$$\tilde{\omega} = \dot{\alpha} \hat{\mathbf{e}}_3^0 + \dot{\beta} \hat{\mathbf{e}}_2' + \dot{\gamma} \hat{\mathbf{e}}_3$$

$$\tilde{\omega} = \dot{\alpha} \begin{pmatrix} -\sin \beta \cos \gamma \\ \sin \beta \sin \gamma \\ \cos \beta \end{pmatrix} + \dot{\beta} \begin{pmatrix} \sin \gamma \\ \cos \gamma \\ 0 \end{pmatrix} + \dot{\gamma} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\tilde{\omega} = \dot{\alpha} \begin{pmatrix} -\sin \beta \cos \gamma \\ \sin \beta \sin \gamma \\ \cos \beta \end{pmatrix} + \dot{\beta} \begin{pmatrix} \sin \gamma \\ \cos \gamma \\ 0 \end{pmatrix} + \dot{\gamma} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\omega}_1 = \dot{\alpha}(-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma$$

$$\tilde{\omega}_2 = \dot{\alpha}(\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma$$

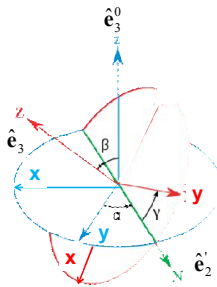
$$\tilde{\omega}_3 = \dot{\alpha} \cos \beta + \dot{\gamma}$$

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$$\tilde{\omega} = \dot{\alpha} \hat{\mathbf{e}}_3^0 + \dot{\beta} \hat{\mathbf{e}}_2' + \dot{\gamma} \hat{\mathbf{e}}_3$$



$$\tilde{\omega} = [\dot{\alpha}(-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma] \hat{\mathbf{e}}_1 + [\dot{\alpha}(\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma] \hat{\mathbf{e}}_2 + [\dot{\alpha} \cos \beta + \dot{\gamma}] \hat{\mathbf{e}}_3$$

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Rotational kinetic energy

$$\begin{aligned} T(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) &= \frac{1}{2} I_1 \tilde{\omega}_1^2 + \frac{1}{2} I_2 \tilde{\omega}_2^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2 \\ &= \frac{1}{2} I_1 [\dot{\alpha}(-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma]^2 \\ &\quad + \frac{1}{2} I_2 [\dot{\alpha}(\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma]^2 \\ &\quad + \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2 \end{aligned}$$

If $I_1 = I_2$:

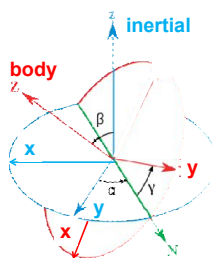
$$T(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2$$

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Transformation between body-fixed and inertial coordinate systems – Euler angles



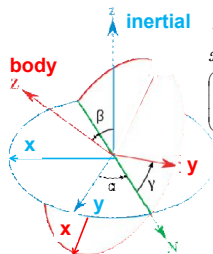
http://en.wikipedia.org/wiki/Euler_angles

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General transformation between rotated coordinates – Euler angles



$$\mathbf{V}' = \mathcal{R}\mathbf{V} = \mathcal{R}_\alpha \mathcal{R}_\beta \mathcal{R}_\gamma \mathbf{V}$$

$$\mathcal{R} = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{pmatrix} \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

http://en.wikipedia.org/wiki/Euler_angles

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