

**PHY 711 Classical Mechanics and
Mathematical Methods**
10-10:50 AM MWF Olin 103

Plan for Lecture 26:

Rigid body rotational motion (Chap. 5)

**1. Use of Euler angles as generalized
coordinates.**

2. Motion of a symmetric top

Begin to discuss membrane vibrations (Chap. 8)

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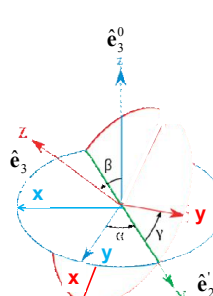
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12 Mon, 9/22/2014	Chap. 3 & 6	Hamiltonian formalism	#11
13 Wed, 9/24/2014	Chap. 3 & 6	Hamiltonian Jacobi transformations	
14 Fri, 9/26/2014	Chap. 4	Small oscillations	Begin Take-Home
15 Mon, 9/29/2014	Chap. 4	Normal modes of motion	Continue Take-Home
16 Wed, 10/01/2014	Chap. 4	Normal modes of motion	Continue Take-Home
17 Fri, 10/03/2014	Chap. 4	Normal modes of motion	Take-Home due
18 Mon, 10/06/2014	Chap. 7	Wave motion	#12
19 Wed, 10/08/2014	Chap. 7	Sturm-Liouville Equations	#13
20 Fri, 10/10/2014	Chap. 7	Sturm-Liouville Equations	#13
21 Mon, 10/13/2014	Chap. 7	Sturm-Liouville Equations	#14
22 Wed, 10/15/2014	Appendix A	Contour integration methods	#15
Fri, 10/17/2014		Fall break -- no class	
23 Mon, 10/20/2014	Appendix A	Fourier transforms	#16
24 Wed, 10/22/2014	Chap. 5	Motion of Rigid Bodies	#17
25 Fri, 10/24/2014	Chap. 5	Motion of Rigid Bodies	#18
26 Mon, 10/27/2014	Chap. 5	Symmetric top in gravitational field	#19
27 Wed, 10/29/2014			
28 Fri, 10/31/2014			

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$$\tilde{\omega} = \dot{\alpha} \hat{e}_3^0 + \dot{\beta} \hat{e}_2' + \dot{\gamma} \hat{e}_3$$

$$\tilde{\omega} = [\dot{\alpha}(-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma] \hat{e}_1 + [\dot{\alpha}(\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma] \hat{e}_2 + [\dot{\alpha} \cos \beta + \dot{\gamma}] \hat{e}_3$$

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Rotational kinetic energy

$$\begin{aligned}
 T(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) &= \frac{1}{2} I_1 \tilde{\omega}_1^2 + \frac{1}{2} I_2 \tilde{\omega}_2^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2 \\
 &= \frac{1}{2} I_1 [\dot{\alpha}(-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma]^2 \\
 &\quad + \frac{1}{2} I_2 [\dot{\alpha}(\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma]^2 \\
 &\quad + \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2
 \end{aligned}$$

If $I_1 = I_2$:

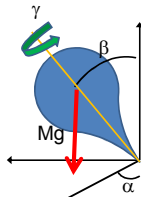
$$T(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2$$

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Motion of a symmetric top under the influence of the torque of gravity:



$$\begin{aligned}
 L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) &= \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \\
 &\quad \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2 - Mgl \cos \beta
 \end{aligned}$$

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$$\begin{aligned}
 L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) &= \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \\
 &\quad \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2 - Mgl \cos \beta
 \end{aligned}$$

Constants of the motion :

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1 \dot{\alpha} \sin^2 \beta + I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}] \cos \beta$$

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]$$

$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + V_{\text{eff}}(\beta)$$

$$L(\beta, \dot{\beta}) = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + \frac{p_\gamma^2}{2I_3} - Mgl \cos \beta$$

$$V_{\text{eff}}(\beta) = \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

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$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

Stable/unstable solutions near $\beta=0$

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Suppose $p_\alpha = p_\gamma$ and $\beta \approx 0$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

$$E' \approx \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_1} \frac{(1 - \frac{1}{2} \beta^2)^2}{\beta^2} + Mgl(1 - \frac{1}{2} \beta^2)$$

$$\approx \frac{1}{2} I_1 \dot{\beta}^2 + \left(\frac{p_\gamma^2}{8I_1} - \frac{Mgl}{2} \right) \beta^2 + Mgl$$

$\Rightarrow$ Stable solution if $p_\gamma \geq \sqrt{4MglI_1}$

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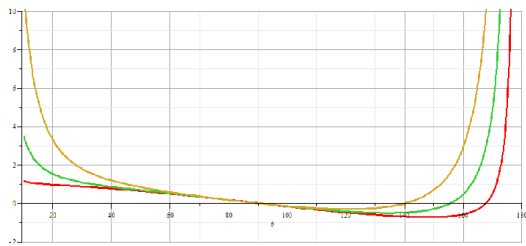
More general case:

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

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More general case:

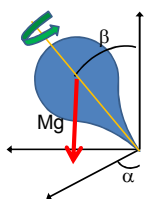
$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2}I_1\dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$



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Constants of the motion :

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3[\dot{\alpha} \cos \beta + \dot{\gamma}]$$

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1 \dot{\alpha} \sin^2 \beta + I_3[\dot{\alpha} \cos \beta + \dot{\gamma}] \cos \beta$$

$$= I_1 \dot{\alpha} \sin^2 \beta + p_\gamma \cos \beta$$

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Elastic media in two or more dimensions --

Review of wave equation in one-dimension -- here $\mu(x,t)$ can describe either a longitudinal or transverse wave.

Traveling wave solutions --

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Note that for any function $f(q)$ or $g(q)$:

$$\mu(x,t) = f(x-ct) + g(x+ct)$$

satisfies the wave equation.

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Initial value problem : $\mu(x,0) = \phi(x)$ and $\frac{\partial \mu}{\partial t}(x,0) = \psi(x)$

then : $\mu(x,0) = \phi(x) = f(x) + g(x)$

$$\frac{\partial \mu}{\partial t}(x,0) = \psi(x) = -c \left(\frac{df(x)}{dx} - \frac{dg(x)}{dx} \right)$$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c} \int \psi(x') dx'$$

For each x , find $f(x)$ and $g(x)$:

$$f(x) = \frac{1}{2} \left(\phi(x) - \frac{1}{c} \int \psi(x') dx' \right)$$

$$g(x) = \frac{1}{2} \left(\phi(x) + \frac{1}{c} \int \psi(x') dx' \right)$$

$$\Rightarrow \mu(x,t) = \frac{1}{2} (\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x') dx'$$

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Standing wave solutions of wave equation:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

with $\mu(0,t) = \mu(L,t) = 0$.

Assume : $\mu(x,t) = \Re(e^{-i\omega t} \rho(x))$

where $\frac{d^2 \rho(x)}{dx^2} + k^2 \rho(x) = 0$ $k = \frac{\omega}{c}$

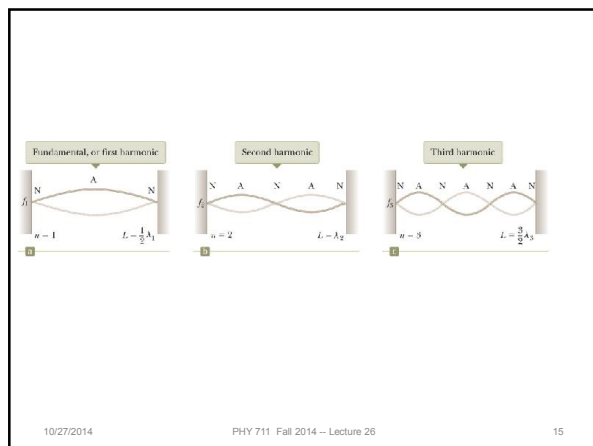
$$\rho_v(x) = A \sin\left(\frac{v\pi x}{L}\right)$$

$$k_v = \frac{v\pi}{L} \quad \omega_v = ck_v$$

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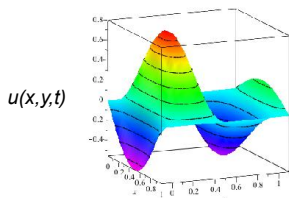


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Extension of ideas to wave motion on a two-dimensional surface – elastic membrane (transverse wave; linear regime).



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Lagrangian density: $\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right)$

$$L = \int \mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) dx dy$$

Hamilton's principle:

$$\delta \int_{t_1}^{t_2} L dt = 0$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial (\partial u / \partial t)} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial (\partial u / \partial x)} - \frac{\partial}{\partial y} \frac{\partial \mathcal{L}}{\partial (\partial u / \partial y)} = 0$$

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Lagrangian density for elastic membrane with constant σ and τ :

$$\mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial t}; x, y, t\right) = \frac{1}{2} \sigma \left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2} \tau (\nabla u)^2$$

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial (\partial u / \partial t)} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial (\partial u / \partial x)} - \frac{\partial}{\partial y} \frac{\partial \mathcal{L}}{\partial (\partial u / \partial y)} = 0$$

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \quad \text{where } c^2 = \frac{\tau}{\sigma}$$

Two - dimensional wave equation :

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0 \quad \text{where } c^2 = \frac{\tau}{\sigma}$$

Standing wave solutions:

$$u(x, y, t) = \Re(e^{-i\omega t} \rho(x, y))$$

$$(\nabla^2 + k^2) \rho(x, y) = 0 \quad \text{where } k = \frac{\omega}{c}$$

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Consider a rectangular boundary:



Clamped boundary conditions:

$$\rho(0, y) = \rho(a, y) = \rho(x, 0) = \rho(x, b) = 0$$

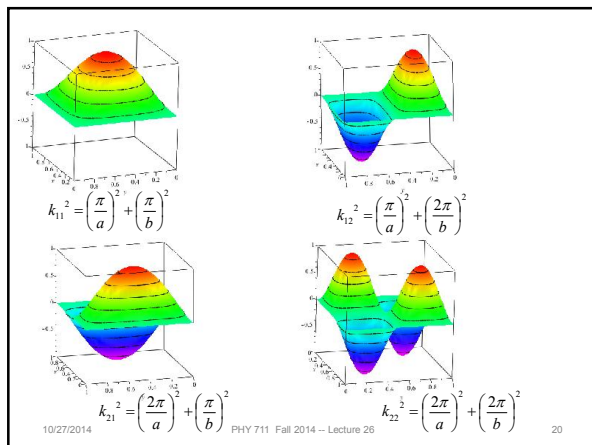
$$\Rightarrow \rho_{mn}(x, y) = A \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad \omega_{mn} = ck_{mn}$$

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More general boundary conditions:

$\tau \nabla u|_b = \kappa u|_b$ represents boundary side constrained with spring

$\tau \nabla u|_b = 0$ represents "free" side

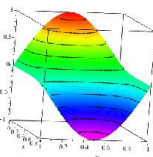
Mixed boundary conditions:

$$\rho(x, 0) = \rho(x, b) = \frac{\partial \rho(0, y)}{\partial x} = \frac{\partial \rho(a, y)}{\partial x} = 0$$

$$\Rightarrow \rho_{mn}(x, y) = A \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$k_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \quad \omega_{mn} = ck_{mn}$$

$$k_{11}^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$



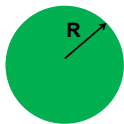
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Consider a circular boundary:

Clamped boundary conditions for $\rho(r, \varphi)$:
 $\rho(R, \varphi) = 0$



$$(\nabla^2 + k^2)\rho(r, \varphi) = 0 \quad \text{where } k = \frac{\omega}{c}$$

In cylindrical coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}$$

Assume: $\rho(r, \varphi) = f(r)\Phi(\varphi)$

Let: $\Phi(\varphi) = e^{im\varphi}$

Note: $\Phi(\varphi) = \Phi(\varphi + 2\pi)$
 $\Rightarrow m = \text{integer}$

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Consider circular boundary -- continued

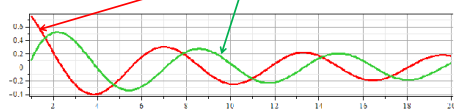
Differential equation for radial function:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + k^2 \right) f(r) = 0$$

$\Rightarrow$ Bessel equation of integer order with transcendental solutions

Cylindrical Bessel function $J_m(z)$

Cylindrical Neumann function $N_m(z)$



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Some properties of Bessel functions

Ascending series: $J_m(z) = \left(\frac{z}{2}\right)^m \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+m)!} \left(\frac{z}{2}\right)^{2j}$

Recursion relations: $J_{m-1}(z) + J_{m+1}(z) = \frac{2m}{z} J_m(z)$

$$J_{m-1}(z) - J_{m+1}(z) = 2 \frac{dJ_m(z)}{dz}$$

Asymptotic form: $J_m(z) \xrightarrow{z \gg 1} \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right)$

Zeros of Bessel functions $J_m(z_{mn}) = 0$

$m = 0$: $z_{0n} = 2.406, 5.520, 8.654, \dots$

$m = 1$: $z_{1n} = 3.832, 7.016, 10.173, \dots$

$m = 2$: $z_{2n} = 5.136, 8.417, 11.620, \dots$

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Some properties of Bessel functions -- continued

Note : It is possible to prove the following

identity for the functions $J_m\left(\frac{z_{mn}}{R}r\right)$:

$$\int_0^R J_m\left(\frac{z_{mn}}{R}r\right) J_m\left(\frac{z_{mn'}}{R}r\right) r dr = \frac{R^2}{2} (J_{m+1}(z_{mn}))^2 \delta_{nn'}$$

Returning to differential equation for radial function :

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + k^2 \right) f(r) = 0$$

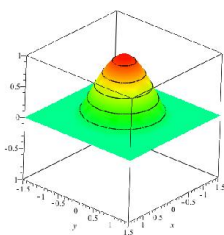
$$\Rightarrow f_{mn}(r) = A J_m\left(\frac{z_{mn}}{R}r\right); \quad k_{mn} = \frac{z_{mn}}{R}$$

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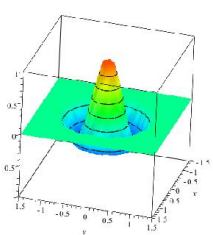
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$$\rho_{01}(r, \varphi) = f_{01}(r) = A J_0\left(\frac{z_{01}}{R}r\right)$$



$$k_{01} = \frac{2.406}{R}$$

$$\rho_{02}(r, \varphi) = f_{02}(r) = A J_0\left(\frac{z_{02}}{R}r\right)$$



$$k_{02} = \frac{5.520}{R}$$

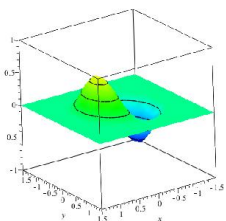
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$$\rho_{11}(r, \varphi) = f_{11}(r) \cos(\varphi)$$

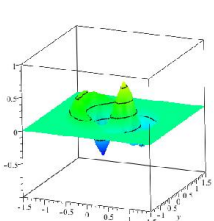
$$= A J_1\left(\frac{z_{11}}{R}r\right) \cos(\varphi)$$



$$k_{11} = \frac{3.832}{R}$$

$$\rho_{12}(r, \varphi) = f_{12}(r) \cos(\varphi)$$

$$= A J_1\left(\frac{z_{12}}{R}r\right) \cos(\varphi)$$



$$k_{12} = \frac{7.016}{R}$$

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