

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF Olin 103**

Plan for Lecture 27:

Introduction to hydrodynamics

- 1. Motivation for topic**
- 2. Newton's laws for fluids**
- 3. Conservation relations**

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13	Wed, 9/24/2014	Chap. 3 & 6	Hamiltonian Jacobi transformations	
14	Fri, 9/26/2014	Chap. 4	Small oscillations	Begin Take-Ho
15	Mon, 9/29/2014	Chap. 4	Normal modes of motion	Continue Take-
16	Wed, 10/01/2014	Chap. 4	Normal modes of motion	Continue Take-
17	Fri, 10/03/2014	Chap. 4	Normal modes of motion	Take-Home du
18	Mon, 10/06/2014	Chap. 7	Wave motion	#12
19	Wed, 10/08/2014	Chap. 7	Sturm-Liouville Equations	#13
20	Fri, 10/10/2014	Chap. 7	Sturm-Liouville Equations	#13
21	Mon, 10/13/2014	Chap. 7	Sturm-Liouville Equations	#14
22	Wed, 10/15/2014	Appendix A	Contour integration methods	#15
	Fri, 10/17/2014		Fall break -- no class	
23	Mon, 10/20/2014	Appendix A	Fourier transforms	#16
24	Wed, 10/22/2014	Chap. 5	Motion of Rigid Bodies	#17
25	Fri, 10/24/2014	Chap. 5	Motion of Rigid Bodies	#18
26	Mon, 10/27/2014	Chap. 5	Symmetric top in gravitational field	#18
27	Wed, 10/29/2014	Chap. 8	Vibrations of membranes	#19
28	Fri, 10/31/2014	Chap. 9	Physics of fluids	#20
29	Mon, 11/03/2014	Chap. 9	Physics of fluids	
30	Wed, 11/05/2014			
31	Fri, 11/07/2014			

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Motivation

1. Natural progression from strings, membranes, fluids; description of 1, 2, and 3 dimensional continua
2. Interesting and technologically important phenomena associated with fluids

Plan

1. Newton's laws for fluids
2. Continuity equation
3. Stress tensor
4. Energy relations
5. Bernoulli's theorem
6. Various examples
7. Sound waves

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Newton's equations for fluids

Use Lagrange formulation; following "particles" of fluid

Variables: Density $\rho(x,y,z,t)$ Pressure $p(x,y,z,t)$ Velocity $\mathbf{v}(x,y,z,t)$

$$m\mathbf{a} = \mathbf{F}$$

$$m \rightarrow \rho dV$$

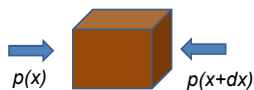
$$\mathbf{a} \rightarrow \frac{d\mathbf{v}}{dt}$$

$$\mathbf{F} \rightarrow \mathbf{F}_{\text{applied}} + \mathbf{F}_{\text{pressure}}$$

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$$\begin{aligned} F_{\text{pressure}}|_x &= (-p(x+dx, y, z) + p(x, y, z)) dydz \\ &= \frac{(-p(x+dx, y, z) + p(x, y, z))}{dx} dx dydz \\ &= -\frac{\partial p}{\partial x} dV \end{aligned}$$

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Newton's equations for fluids -- continued

$$m\mathbf{a} = \mathbf{F}_{\text{applied}} + \mathbf{F}_{\text{pressure}}$$

$$\rho dV \frac{d\mathbf{v}}{dt} = \mathbf{f}_{\text{applied}} \rho dV - (\nabla p) dV$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

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Detailed analysis of acceleration term:

$$\mathbf{v} = \mathbf{v}(x, y, z, t)$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x} \frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y} \frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z} \frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x} v_x + \frac{\partial \mathbf{v}}{\partial y} v_y + \frac{\partial \mathbf{v}}{\partial z} v_z + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}$$

Note that :

$$\frac{\partial \mathbf{v}}{\partial x} v_x + \frac{\partial \mathbf{v}}{\partial y} v_y + \frac{\partial \mathbf{v}}{\partial z} v_z = \nabla \left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) - \mathbf{v} \times (\nabla \times \mathbf{v})$$

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Newton's equations for fluids -- continued

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

$$\rho \left(\nabla \left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) + \frac{\partial \mathbf{v}}{\partial t} \right) = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

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Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla \times \mathbf{v}) = 0$ "irrotational flow"

$$\Rightarrow \mathbf{v} = -\nabla \Phi$$

2. $\mathbf{f}_{\text{applied}} = -\nabla U$ conservative applied force

3. $\rho = (\text{constant})$ incompressible fluid

$$\frac{\partial (-\nabla \Phi)}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla \left(\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} \right) = 0$$

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Bernoulli's integral of Euler's equation

$$\nabla \left(\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} \right) = 0$$

Integrating over space:

$$\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} = C(t)$$

where $\mathbf{v} = -\nabla \Phi(\mathbf{r}, t) = -\nabla(\Phi(\mathbf{r}, t) + C'(t))$

$$\Rightarrow \frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} = 0 \quad \text{Bernoulli's theorem}$$

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Examples of Bernoulli's theorem

$$\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} = 0$$

Modified form; assuming $\frac{\partial \Phi}{\partial t} = 0$

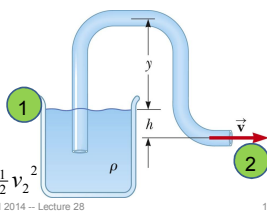
$$\frac{p}{\rho} + U + \frac{1}{2} v^2 = \text{constant}$$

$$p_1 = p_2 = p_{atm}$$

$$U_1 - U_2 = gh$$

$$v_1 \approx 0$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2} v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2} v_2^2$$

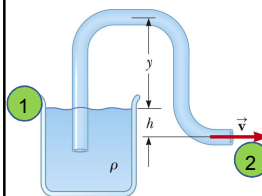


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Examples of Bernoulli's theorem -- continued



$$p_1 = p_2 = p_{atm}$$

$$U_1 - U_2 = gh$$

$$v_1 \approx 0$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2} v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2} v_2^2$$

$$v_2 \approx \sqrt{2gh}$$

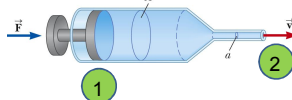
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Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \frac{1}{2} v^2 = \text{constant}$$



$$p_1 = \frac{F}{A} + p_{atm} \quad p_2 = p_{atm}$$

$$U_1 = U_2$$

$$v_1 A = v_2 a \quad \text{continuity equation}$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2} v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2} v_2^2$$

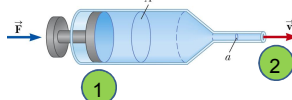
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Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \frac{1}{2} v^2 = \text{constant}$$



$$\frac{2F}{A} = v_2^2 \left(1 - \left(\frac{a}{A} \right)^2 \right)$$

$$v_2 = \sqrt{\frac{2F/A}{1 - \left(\frac{a}{A} \right)^2}}$$

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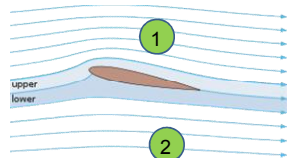
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Examples of Bernoulli's theorem -- continued

Approximate explanation of airplane lift

Cross section view of airplane wing

http://en.wikipedia.org/wiki/Lift_%28force%29


$$U_1 \approx U_2$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2} v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2} v_2^2$$

$$p_2 - p_1 = \frac{1}{2} (v_1^2 - v_2^2)$$

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Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

Consider: $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\nabla \rho) \cdot \mathbf{v}$

$$\Rightarrow \frac{d\rho}{dt} + \rho(\nabla \cdot \mathbf{v}) = 0 \quad \text{alternative form of continuity equation}$$

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Some details on the velocity potential

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

For incompressible fluid: $\rho = (\text{constant})$

$$\Rightarrow \nabla \cdot \mathbf{v} = 0$$

Irrotational flow: $\nabla \times \mathbf{v} = 0 \quad \Rightarrow \mathbf{v} = -\nabla \Phi$

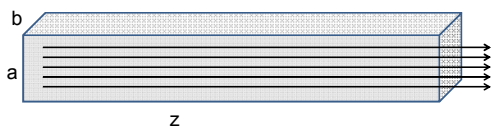
$$\Rightarrow \nabla^2 \Phi = 0$$

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Example – uniform flow



$$\nabla^2 \Phi = 0$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Possible solution:

$$\Phi = -v_0 z$$

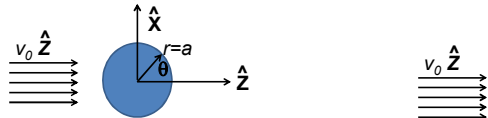
$$\mathbf{v} = -\nabla \Phi = v_0 \hat{\mathbf{z}}$$

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Example – flow around a long cylinder (oriented in the Y direction)



$$\nabla^2 \Phi = 0$$

$$\left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 0$$

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Laplace equation in cylindrical coordinates

(r, θ , defined in x - z plane; y representing cylinder axis)

$$\nabla^2 \Phi = 0 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r, \theta, y) = \Phi(r, \theta)$$

From boundary condition : $v_z(r \rightarrow \infty) = v_0$

$$\frac{\partial \Phi}{\partial z}(r \rightarrow \infty) = -v_0 \quad \Rightarrow \quad \Phi(r \rightarrow \infty, \theta) = -v_0 r \cos \theta$$

$$\text{Note that : } \frac{\partial^2 \cos \theta}{\partial \theta^2} = -\cos \theta$$

$$\text{Guess form : } \Phi(r, \theta) = f(r) \cos \theta$$

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Necessary equation for radial function

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial f}{\partial r} - \frac{1}{r^2} f = 0$$

$$f(r) = Ar + \frac{B}{r} \quad \text{where } A, B \text{ are constants}$$

Boundary condition on cylinder surface :

$$\left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 0$$

$$\frac{df}{dr}(r=a) = 0 = A - \frac{B}{a^2}$$

$$\Rightarrow B = Aa^2$$

Boundary condition at ∞ : $\Rightarrow A = -v_0$

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$$\Phi(r, \theta) = -v_0 \left(r + \frac{a^2}{r} \right) \cos \theta$$

$$v_r = -\frac{\partial \Phi}{\partial r} = v_0 \left(1 - \frac{a^2}{r^2} \right) \cos \theta$$

$$v_\theta = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta} = v_0 \left(1 + \frac{a^2}{r^2} \right) \sin \theta$$

For homework; consider similar boundary value problem for
a spherical obstruction

Laplacian in spherical polar coordinates :

$$\nabla^2 \Phi = 0 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2}$$

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