PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 2:

1. Brief comment on quiz
2. Particle interactions
3. Notion of center of mass reference fame
4. Introduction to scattering theory

Comment on quiz questions

1. $g(t) = \int_0^t (x^2 + t)\,dx \qquad \frac{dg}{dt} = \int_0^t \frac{d(x^2+t)}{dt} + (x^2+t)\Big|_{x=t} = t^2 + 2t$

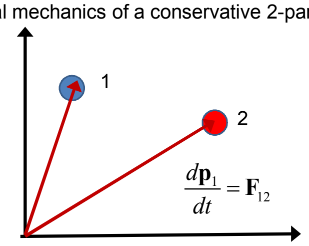
2. Evaluate the integral $\oint \frac{dz}{z}$ for a closed contour about the origin.

$$\oint \frac{dz}{z} = 2\pi i$$

3. $\frac{df}{dx} = f \qquad \Rightarrow f(x) = Ae^x$

4. $\sum_{n=1}^{N} a^n = \frac{a - a^{N+1}}{1-a}$

Classical mechanics of a conservative 2-particle system.



$$\frac{d\mathbf{p}_1}{dt} = \mathbf{F}_{12} \qquad \frac{d\mathbf{p}_2}{dt} = \mathbf{F}_{21}$$

$$\mathbf{F}_{12} = -\nabla_1 V(\mathbf{r}_1 - \mathbf{r}_2) \qquad \Rightarrow E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

Relationship between center of mass and laboratory frames of reference

Definition of center of mass $\mathbf{R}_{CM}$

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = (m_1+m_2)\mathbf{R}_{CM}$$

$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = (m_1+m_2)\dot{\mathbf{R}}_{CM} = (m_1+m_2)\mathbf{V}_{CM}$$

$$E = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + V(\mathbf{r}_1-\mathbf{r}_2)$$

$$= \frac{1}{2}(m_1+m_2)V_{CM}^2 + \frac{1}{2}\mu|\mathbf{v}_1-\mathbf{v}_2|^2 + V(\mathbf{r}_1-\mathbf{r}_2)$$

where: $\mu \equiv \dfrac{m_1m_2}{m_1+m_2}$

8/29/2014 PHY 711 Fall 2014 -- Lecture 2 4

Classical mechanics of a conservative 2-particle system -- continued

$$E = \frac{1}{2}(m_1+m_2)V_{CM}^2 + \frac{1}{2}\mu|\mathbf{v}_1-\mathbf{v}_2|^2 + V(\mathbf{r}_1-\mathbf{r}_2)$$

For central potentials: $V(\mathbf{r}_1-\mathbf{r}_2) = V(|\mathbf{r}_1-\mathbf{r}_2|) \equiv V(r_{12})$

Relative angular momentum is also conserved:

$$\mathbf{L}_{12} \equiv \mathbf{r}_{12}\times\mu\mathbf{v}_{12}$$

$$E = \frac{1}{2}(m_1+m_2)V_{CM}^2 + \frac{1}{2}\mu v_{12}^2 + \frac{L_{12}^2}{2\mu r_{12}^2} + V(r_{12})$$

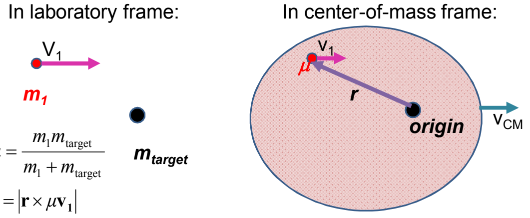
Simpler notation:

$$E = \frac{1}{2}(m_1+m_2)V_{CM}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

8/29/2014 PHY 711 Fall 2014 -- Lecture 2 5

Note: The following analysis will be carried out in the center of mass frame of reference.

In laboratory frame: In center-of-mass frame:



$$\mu = \frac{m_1 m_{\text{target}}}{m_1 + m_{\text{target}}}$$

$$\ell = |\mathbf{r}\times\mu\mathbf{v}_1|$$

Also note: We are assuming that the interaction between particle and target $V(r)$ conserves energy and angular momentum.

8/29/2014 PHY 711 Fall 2014 -- Lecture 2 6

Scattering theory:

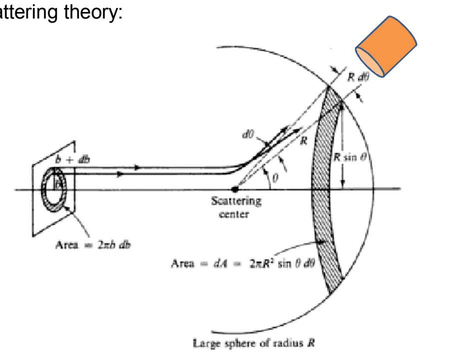
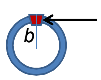


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

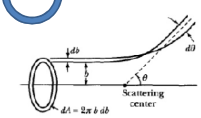
Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

$= \text{Area of incident beam that is scattered into detector at angle } \theta$



$d\varphi b db$

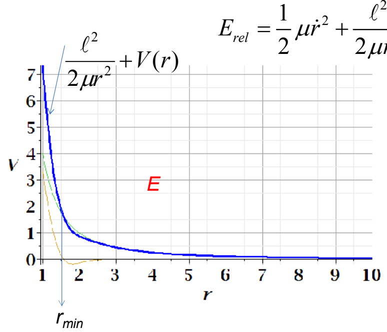


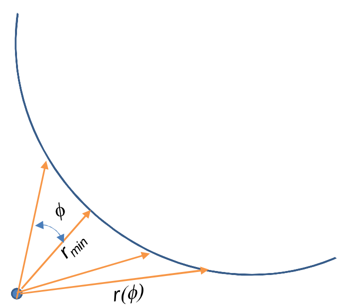
$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

In center of mass reference frame:

$$E_{rel} = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$





Conservation of angular momentum:

$$\ell = \mu r^2 \left(\frac{d\phi}{dt}\right)$$

Transformation of trajectory variables:

$$r(t) \Leftrightarrow r(\phi)$$

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{dr}{d\phi}\frac{\ell}{\mu r^2}$$

Conservation of energy in the center of mass frame:

$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$= \frac{1}{2}\mu\left(\frac{dr}{d\phi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$\Rightarrow E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$= \frac{1}{2}\mu\left(\frac{dr}{d\phi}\frac{\ell}{\mu r^2}\right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Solving for $r(\phi) \Leftrightarrow \phi(r)$

$$\left(\frac{dr}{d\phi}\right)^2 = \left(\frac{2\mu r^4}{\ell^2}\right)\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)$$

$$d\phi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\phi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

Further simplification at large separation:

$$\ell = \mu v_\infty b$$

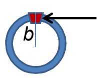
$$E = \frac{1}{2}\mu v_\infty^2$$

$$\Rightarrow \ell = \sqrt{2\mu E}b$$

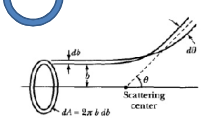
Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

= Area of incident beam that is scattered into detector at angle θ



$d\varphi b db$



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\varphi\, b\, db}{d\varphi \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Figure from Marion & Thorton, Classical Dynamics

When the dust clears:

$$d\phi = dr\left(\frac{\ell / r^2}{\sqrt{2\mu\left(E - \frac{\ell^2}{2\mu r^2} - V(r)\right)}}\right)$$

$$d\phi = dr\left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}}\right)$$

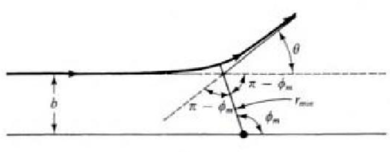
$$\Rightarrow \phi(b, E)$$

$$\int_0^{\phi_{\max}} d\phi = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1-\frac{b^2}{r^2}-\frac{V(r)}{E}}} \right)$$

where :

$$1-\frac{b^2}{r_{\min}^2}-\frac{V(r_{\min})}{E}=0$$

Relationship between ϕ_{max} and θ:



$$2(\pi-\phi_{\max})+\theta=\pi$$

$$\Rightarrow \phi_{\max}=\frac{\pi}{2}-\frac{\theta}{2}$$

$$\phi_{\max}=\frac{\pi}{2}-\frac{\theta}{2}=\int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1-\frac{b^2}{r^2}-\frac{V(r)}{E}}} \right)$$

$$\theta=\pi-2b\int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1-\frac{b^2}{r^2}-\frac{V(r)}{E}}} \right)$$

$$\theta=\pi-2b\int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1-b^2u^2-\frac{V(1/u)}{E}}} \right)$$

Scattering angle equation :

$$\theta = \pi - 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

Rutherford scattering example :

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \qquad \frac{1}{r_{\min}} = \frac{1}{b}\left(-\frac{\kappa}{2b} + \sqrt{\left(\frac{\kappa}{2b}\right)^2 + 1} \right)$$

$$\theta = \pi - 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \kappa u}} \right) = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

Rutherford scattering continued :

$$\theta = 2\sin^{-1}\left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

$$\frac{2b}{\kappa} = \left| \frac{\cos(\theta/2)}{\sin(\theta/2)} \right|$$

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right| = \frac{\kappa^2}{16} \frac{1}{\sin^4(\theta/2)}$$