PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 30: Chap. 9 of F&W

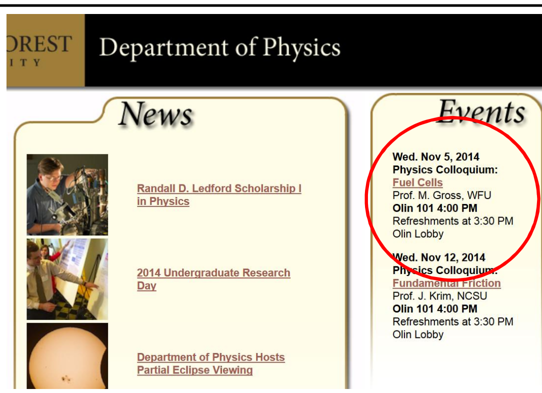
Wave equation for sound in linear approximation

1. **Standing wave solutions**
2. **Inhomogeneous equations**

11/05/2014 PHY 711 Fall 2014 -- Lecture 30 1

17	Fri, 10/03/2014	Chap. 4	Normal modes of motion	Take-Home due
18	Mon, 10/06/2014	Chap. 7	Wave motion	#12
19	Wed, 10/08/2014	Chap. 7	Sturm-Liouville Equations	#13
20	Fri, 10/10/2014	Chap. 7	Sturm-Liouville Equations	#13
21	Mon, 10/13/2014	Chap. 7	Sturm-Liouville Equations	#14
22	Wed, 10/15/2014	Appendix A	Contour integration methods	#15
	Fri, 10/17/2014		Fall break -- no class	
23	Mon, 10/20/2014	Appendix A	Fourier transforms	#16
24	Wed, 10/22/2014	Chap. 5	Motion of Rigid Bodies	#17
25	Fri, 10/24/2014	Chap. 5	Motion of Rigid Bodies	#18
26	Mon, 10/27/2014	Chap. 5	Symmetric top in gravitational field	#18
27	Wed, 10/29/2014	Chap. 8	Vibrations of membranes	#19
28	Fri, 10/31/2014	Chap. 9	Physics of fluids	#20
29	Mon, 11/03/2014	Chap. 9	Physics of fluids	#21
30	Wed, 11/05/2014	Chap. 9	Sound waves	
31	Fri, 11/07/2014	Chap. 9	Sound waves	Begin Take-Home
32	Mon, 11/10/2014			Continue Take-Home
33	Wed, 11/12/2014			Continue Take-Home
34	Fri, 11/14/2014			Continue Take-Home
35	Mon, 11/17/2014			Take-Home due
36	Wed, 11/19/2014			

11/05/2014 PHY 711 Fall 2014 -- Lecture 30 2



11/05/2014 PHY 711 Fall 2014 -- Lecture 30 3

WFU Physics Colloquium

TITLE: Solid Oxide Fuel Cells: Energy, Materials, Composites, and Design

SPEAKER: Professor Michael Gross,

Department of Chemistry
Wake Forest University

TIME: Wednesday November 5, 2014 at 4:00 PM

PLACE: Room 101 Olin Physical Laboratory

Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

ABSTRACT

Solid Oxide Fuel Cell (SOFC) technology has the ability to efficiently convert a wide range of chemical energy sources directly into electrical energy. Basic SOFC operating mechanisms will be introduced followed by several areas of research being conducted in the Gross research group. Areas of study include the structure and performance of solid state materials as well as the design of multifunctional composites to achieve increased SOFC electrode performance, stability, and fuel flexibility. In addition to individual material synthesis and development, computational models are utilized to predict electrode composite morphologies and designs that result in optimized electrode performance.

Application of fluid equations to the case of air in equilibrium plus small perturbation

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$

Near equilibrium :

$$\rho = \rho_0 + \delta\rho$$
$$p = p_0 + \delta p$$
$$\mathbf{v} = 0 + \delta\mathbf{v}$$
$$\mathbf{f}_{applied} = 0$$

Equations to lowest order in perturbation :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} \quad\Rightarrow\quad \frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \quad\Rightarrow\quad \frac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot\delta\mathbf{v} = 0$$

In terms of the velocity potential :

$$\delta\mathbf{v} = -\nabla\Phi$$

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0} \quad\Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot\delta\mathbf{v} = 0 \quad\Rightarrow \frac{\partial \delta\rho}{\partial t} - \rho_0\nabla^2\Phi = 0$$

Expressing pressure in terms of the density:

$p = p(s,\rho) = p_0 + \delta p$ where s denotes the (constant) entropy

$p_0 = p(s,\rho_0)$

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv c^2 \delta\rho$$

$$\nabla\left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0 \quad \Rightarrow -\frac{\partial \Phi}{\partial t} + c^2 \frac{\delta\rho}{\rho_0} = (\text{constant})$$

$$\Rightarrow -\frac{\partial^2 \Phi}{\partial t^2} + \frac{c^2}{\rho_0}\frac{\partial \delta\rho}{\partial t} = 0$$

$$\frac{\partial \delta\rho}{\partial t} - \rho_0 \nabla^2 \Phi = 0 \quad \Rightarrow \frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Wave equation for air:

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$

$\mathbf{v} = -\nabla\Phi$

Note that, we also have:

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c^2 \nabla^2 \delta\rho = 0$$

$$\frac{\partial^2 \delta p}{\partial t^2} - c^2 \nabla^2 \delta p = 0$$

Boundary values:

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta\mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla\Phi$$

Free surface:

$$\delta p = 0 \quad \Rightarrow \rho_0 \frac{\partial \Phi}{\partial t} = 0$$

Analysis of wave velocity in an ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

Equation of state for ideal gas:

$$pV = NkT \qquad N = \frac{M}{M_0}$$

$$p = \frac{M}{V}\frac{k}{M_0}T = \rho \frac{k}{M_0} T$$

$k = 1.38 \times 10^{-23}\, J/k$

M_0 = average mass of each molecule

Estimate of the speed of sound:

Isentropic or adiabatic equation of state:

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \qquad \Rightarrow \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

Linearized speed of sound

$$c_0{}^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s,p_0,\rho_0} = \frac{p_0\gamma}{\rho_0}$$

$$c_0{}^2 \approx \frac{1.5 \cdot 1.013 \times 10^5\, Pa}{1.3 kg / m^3} \qquad c_0 \approx 340 \text{ m/s}$$

Density dependence of speed of sound for ideal gas:

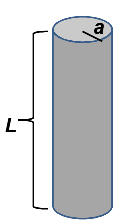
$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$c^2 = \frac{p_0\gamma}{\rho_0}\frac{p/p_0}{\rho/\rho_0} = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$$

where $c_0^2 = \frac{p_0\gamma}{\rho_0}$

Time harmonic standing waves in a pipe



$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Boundary values :

At fixed surface : $\hat{\mathbf{n}} \cdot \nabla \Phi = 0$

At free surface : $\frac{\partial \Phi}{\partial t} = 0$

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \qquad \text{Define : } \quad k \equiv \frac{\omega}{c}$$

In cylindrical coordinates :

$$\Phi(r,\varphi,z,t) = R(r)F(\varphi)Z(z)e^{-i\omega t} \equiv R(r)F(\varphi)Z(z)e^{-ikct}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\Phi(r,\varphi,z,t) = 0$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\Phi(r,\varphi,z,t) = 0$$

$$\Phi(r,\varphi,z,t) = R(r)F(\varphi)Z(z)e^{-i\omega t}$$

$$F(\varphi) = e^{im\varphi};\ F(\varphi) = F(\varphi + 2\pi N) \Rightarrow m = \text{integer}$$

$$Z(z) = e^{i\alpha z};\ \alpha = \text{real plus other restrictions}$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - \alpha^2 + k^2\right)R(r) = 0$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - \alpha^2 + k^2\right)R(r) = 0$$

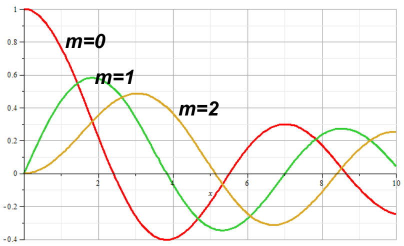
For $k^2 \geq \alpha^2$ define $\kappa^2 \equiv k^2 - \alpha^2$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + \kappa^2\right)R(r) = 0$$

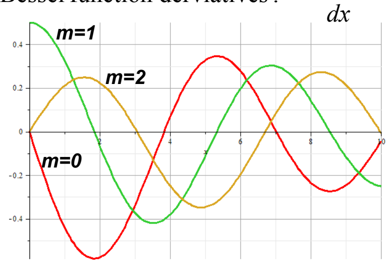
Cylinder surface boundary conditions : $\left.\frac{dR}{dr}\right|_{r=a} = 0$

$\Rightarrow R(r) = J_m(\kappa r)$ where for $\frac{dJ_m(x'_{mn})}{dx} = 0,\ \kappa_{mn} = \frac{x'_{mn}}{a}$

Bessel functions : $J_m(x)$



Bessel function derviatives : $\frac{dJ_m(x)}{dx}$



Zeros of derivatives: m=0: 0.00000, 3.83171, 7.01559
m=1: 1.84118, 5.33144, 8.53632
m=2: 3.05424, 6.70613, 9.96947

Boundary condition for z=0, z=L:

For open - open pipe :

$$Z(0) = Z(L) = 0 \quad \Rightarrow Z(z) = \sin\left(\frac{p\pi z}{L}\right)$$

$$\Rightarrow \alpha_p = \frac{p\pi}{L}, \qquad p = 1,2,3...$$

Resonant frequencies :

$$\frac{\omega^2}{c^2} = k^2 = \kappa_{mn}^2 + \alpha_p^2$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2$$

Example

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2 = \left(\frac{\pi p}{L}\right)^2\left(1+\left(\frac{L}{a}\right)^2\left(\frac{x'_{mn}}{\pi p}\right)^2\right)$$

$$\pi p = 3.14, 6.28, 9.42....$$
$$x'_{mn} = 0.00, 1.84, 3.05$$

Alternate boundary condition for z=0, z=L:

For open - closed pipe :

$$\frac{dZ(0)}{dz} = Z(L) = 0 \quad \Rightarrow Z(z) = \cos\left(\frac{(2p+1)\pi z}{2L}\right)$$

$$\Rightarrow \alpha_p = \frac{(2p+1)\pi}{2L}, \quad p = 0,1,2,3...$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi(2p+1)}{2L}\right)^2$$

Other solutions to wave equation:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

Plane wave solution :

$$\Phi(\mathbf{r},t) = Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Solution in terms of Green's function :

$$\Phi(\mathbf{r},t) = \int d^3r' \int dt' \, G(\mathbf{r}-\mathbf{r}', t-t') f(\mathbf{r}', t')$$

where

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Wave equation with source -- continued:

We can show that :

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{\delta\left(t' - \left(t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Derivation of Green's function for wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Recall that

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')} d\omega$$

Derivation of Green's function for wave equation -- continued

$$\text{Define}: \quad \tilde{G}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} G(\mathbf{r},t) e^{i\omega t} dt$$

$$G(\mathbf{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{G}(\mathbf{r},\omega) e^{-i\omega t} d\omega$$

$\tilde{G}(\mathbf{r},\omega)$ must satisfy :

$$(\nabla^2 + k^2)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}') \quad \text{where } k^2 = \frac{\omega^2}{c^2}$$

11/11/2013 PHY 711 Fall 2013 -- Lecture 29 25

Derivation of Green's function for wave equation -- continued

$$(\nabla^2 + k^2)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

Solution assuming isotropy in $\mathbf{r}-\mathbf{r}'$:

$$\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Check -- Define $R \equiv |\mathbf{r}-\mathbf{r}'|$ and for $R > 0$:

$$(\nabla^2 + k^2)\tilde{G}(R,\omega) = \frac{1}{R}\frac{d^2}{dR^2}\left(R\tilde{G}(R,\omega)\right) + k^2\tilde{G}(R,\omega) = 0$$

11/11/2013 PHY 711 Fall 2013 -- Lecture 29 26

Derivation of Green's function for wave equation -- continued

For $R > 0$:

$$(\nabla^2 + k^2)\tilde{G}(R,\omega) = \frac{1}{R}\frac{d^2}{dR^2}\left(R\tilde{G}(R,\omega)\right) + k^2\tilde{G}(R,\omega) = 0$$

$$\frac{d^2}{dR^2}\left(R\tilde{G}(R,\omega)\right) + k^2\left(R\tilde{G}(R,\omega)\right) = 0$$

$$\left(R\tilde{G}(R,\omega)\right) = A\, e^{ikR} + Be^{-ikR}$$

$$\Rightarrow \tilde{G}(R,\omega) = A\frac{e^{ikR}}{R} + B\frac{e^{-ikR}}{R}$$

11/11/2013 PHY 711 Fall 2013 -- Lecture 29 27

Derivation of Green's function for wave equation – continued
need to find A and B.

$$\text{Note that}: \quad \nabla^2 \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = -\delta(\mathbf{r}-\mathbf{r}')$$

$$\Rightarrow A = B = \frac{1}{4\pi}$$

$$\tilde{G}(R,\omega) = \frac{e^{\pm ikR}}{4\pi R}$$

Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(\mathbf{r}-\mathbf{r}',\omega) e^{-i\omega(t-t')} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} e^{-i\omega(t-t')} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} e^{-i\omega(t-t')} d\omega$$

Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} e^{-i\omega(t-t')} d\omega$$

$$\text{Noting that} \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega u} d\omega = \delta(u)$$

$$\Rightarrow G(\mathbf{r}-\mathbf{r}',t-t') = \frac{\delta\left(t-\left(t' \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

For time harmonic forcing term we can use the corresponding Green's function:

$$\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)=\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

In fact, this Green's function is appropriate for boundary conditions at infinity. For surface boundary conditions where we know the boundary values or their gradients, the Green's function must be modified.

Green's theorem

Consider two functions $h(\mathbf{r})$ and $g(\mathbf{r})$

Note that : $\int_V \left(h\nabla^2 g - g\nabla^2 h\right)d^3r = \oint_S \left(h\nabla g - g\nabla h\right)\cdot\hat{\mathbf{n}}\,d^2r$

$$\nabla^2\widetilde{\Phi}+k^2\widetilde{\Phi}=-\widetilde{f}(\mathbf{r},\omega)$$

$$\left(\nabla^2+k^2\right)\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega)=-\delta(\mathbf{r}-\mathbf{r}')$$

$$h\leftrightarrow\widetilde{\Phi};\qquad g\leftrightarrow\widetilde{G}$$

$$\int_V\left(\widetilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}')-\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)f(\mathbf{r},\omega)\right)d^3r=$$

$$\oint_S\left(\widetilde{\Phi}(\mathbf{r},\omega)\nabla\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)-\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\widetilde{\Phi}(\mathbf{r},\omega)\right)\cdot\hat{\mathbf{n}}\,d^2r$$

$$\widetilde{\Phi}(\mathbf{r},\omega)=\int_V\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)f(\mathbf{r}',\omega)d^3r'+$$

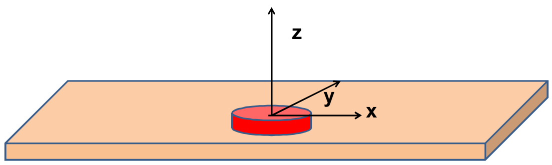
$$\oint_S\left(\widetilde{\Phi}(\mathbf{r}',\omega)\nabla\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)-\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\widetilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}\,d^2r'$$

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Example :

$f(\mathbf{r},t) \Rightarrow$ time harmonic piston of radius a, amplitude $\varepsilon\hat{\mathbf{z}}$
can be represented as boundary value of $\Phi(\mathbf{r},t)$



Treatment of boundary values for time-harmonic force:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\tilde{f}(\mathbf{r}',\omega)d^3r' +$$
$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla'\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla'\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

Boundary values for our example :

$$\left(\frac{\partial\tilde{\Phi}}{\partial z}\right)_{z=0} = \begin{cases} 0 & \text{for } x^2+y^2 > a^2 \\ i\omega\varepsilon a & \text{for } x^2+y^2 < a^2 \end{cases}$$

Note : Need Green's function with vanishing gradient at $z = 0$:

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z > 0$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\,z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'dy'$$

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z > 0$$

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)_{z'=0} = \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}; \quad z > 0$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\, z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'dy'$$

$$= -i\omega\varepsilon a\int_0^a r'dr'\int_0^{2\pi} d\phi' \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}$$

Integration domain : $x' = r'\cos\phi'$

$y' = r'\sin\phi'$

For $r >> a$; $|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\phi = \frac{\pi}{2}$

$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' = r - r'\sin\theta\sin\phi'$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{i\omega\varepsilon a}{2\pi}\frac{e^{ikr}}{r}\int_0^a r'dr'\int_0^{2\pi} d\phi' e^{-ikr'\sin\theta\sin\phi'}$$

Note that : $\frac{1}{2\pi}\int_0^{2\pi} d\phi' e^{-iu\sin\phi'} = J_0(u)$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a\frac{e^{ikr}}{r}\int_0^a r'dr' J_0(kr'\sin\theta)$$

$$\int_0^w udu J_0(u) = wJ_1(w)$$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a^3\frac{e^{ikr}}{r}\frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

Energy flux : $\mathbf{j}_e = \delta\mathbf{v}p$

Taking time average : $\langle\mathbf{j}_e\rangle = \frac{1}{2}\Re(\delta\mathbf{v}p^*)$

$$= \frac{1}{2}\rho_0\Re\left((-\nabla\Phi)(-i\omega\Phi)^*\right)$$

Time averaged power per solid angle :

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \langle\mathbf{j}_e\rangle\cdot\hat{\mathbf{r}}r^2 = \frac{1}{2}\rho_0\varepsilon^2c^3k^4a^6\left|\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right|^2$$

Time averaged power per solid angle :

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2} \rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka \sin\theta)}{ka \sin\theta} \right|^2$$