

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF Olin 103**

Plan for Lecture 35:

Chapter 11 in F & W:

Heat conduction

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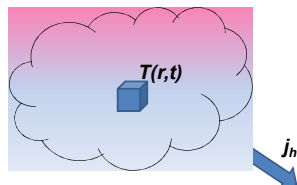
25	Fri, 10/24/2014	Chap. 5	Motion of Rigid Bodies	#18
26	Mon, 10/27/2014	Chap. 5	Symmetric top in gravitational field	#18
27	Wed, 10/29/2014	Chap. 8	Vibrations of membranes	#19
28	Fri, 10/31/2014	Chap. 9	Physics of fluids	#20
29	Mon, 11/03/2014	Chap. 9	Physics of fluids	#21
30	Wed, 11/05/2014	Chap. 9	Sound waves	
31	Fri, 11/07/2014	Chap. 9	Sound waves	Begin Take-Home
32	Mon, 11/10/2014	Chap. 9	Non-linear effects	Continue Take-Home
33	Wed, 11/12/2014	Chap. 10	Surface waves in fluids	Continue Take-Home
34	Fri, 11/14/2014	Chap. 10	Surface waves in fluids	Continue Take-Home
35	Mon, 11/17/2014	Chap. 11	Heat Conduction	Take-Home due #22
36	Wed, 11/19/2014	Chap. 11	Heat Conduction	
37	Fri, 11/21/2014			
38	Mon, 11/24/2014			
	Wed, 11/26/2014		Thanksgiving Holiday	
	Fri, 11/28/2014		Thanksgiving Holiday	
39	Mon, 12/01/2014			
	Wed, 12/03/2014		Student presentations I	
	Fri, 12/05/2014		Student presentations II	
	Mon, 12/08/2014		Begin Take-home final	

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Conduction of heat



Enthalpy of a system at constant pressure p
 non uniform temperature $T(\mathbf{r}, t)$
 mass density ρ and heat capacity c_p

$$H = \int_V \rho c_p (T(\mathbf{r}, t) - T_0) d^3r + H_0(T_0, p)$$

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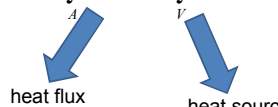
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Conduction of heat -- continued

$$H = \int_V \rho c_p (T(\mathbf{r}, t) - T_0) d^3r + H_0(T_0, p)$$

Time rate of change of enthalpy:

$$\frac{dH}{dt} = \int_V \rho c_p \frac{\partial T(\mathbf{r}, t)}{\partial t} d^3r = - \int_A \mathbf{j}_h \cdot d\mathbf{A} + \int_V \rho \dot{q} d^3r$$



$$\rho c_p \frac{\partial T(\mathbf{r}, t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

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Conduction of heat -- continued

$$\rho c_p \frac{\partial T(\mathbf{r}, t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

Empirically: $\mathbf{j}_h = -k_{th} \nabla T(\mathbf{r}, t)$

$$\Rightarrow \frac{\partial T(\mathbf{r}, t)}{\partial t} = \kappa \nabla^2 T(\mathbf{r}, t) + \frac{\dot{q}}{c_p}$$

$$\kappa \equiv \frac{k_{th}}{\rho c_p}$$

Typical values (m²/s)

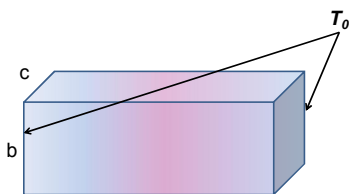
Air 2×10^{-5}
 Water 1×10^{-7}
 Copper 1×10^{-4}

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Boundary value problems for heat conduction



$$\frac{\partial T(\mathbf{r}, t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r}, t) = \frac{\dot{q}}{c_p}$$

Without source term: $\frac{\partial T(\mathbf{r}, t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r}, t) = 0$

Example with boundary values: $T(0, y, z, t) = T(a, y, z, t) = T_0$

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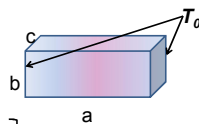
Boundary value problems for heat conduction

$$\frac{\partial T(\mathbf{r}, t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r}, t) = 0$$

$$T(0, y, z, t) = T(a, y, z, t) = T_0$$

$$\frac{\partial T(x, 0, z, t)}{\partial y} = \frac{\partial T(x, b, z, t)}{\partial y} = 0$$

$$\frac{\partial T(x, y, 0, t)}{\partial z} = \frac{\partial T(x, y, c, t)}{\partial z} = 0$$



Assuming thermally insulated boundaries

Separation of variables: $T(x, y, z, t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$

$$\text{Let } \frac{d^2 X}{dx^2} = -\alpha^2 X \quad \frac{d^2 Y}{dy^2} = -\beta^2 Y \quad \frac{d^2 Z}{dz^2} = -\gamma^2 Z$$

$$\Rightarrow -\lambda + \kappa(\alpha^2 + \beta^2 + \gamma^2) = 0$$

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Boundary value problems for heat conduction

$$T(x, y, z, t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$$

$$X(0) = X(a) = 0 \quad \Rightarrow X(x) = \sin\left(\frac{m\pi x}{a}\right)$$

$$\frac{dY(0)}{dy} = \frac{dY(b)}{dy} = 0 \quad \Rightarrow Y(y) = \cos\left(\frac{n\pi y}{b}\right)$$

$$\frac{dZ(0)}{dz} = \frac{dZ(c)}{dz} = 0 \quad \Rightarrow Z(z) = \cos\left(\frac{p\pi z}{c}\right)$$

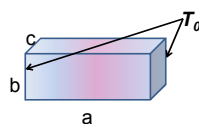
$$-\lambda_{nmp} + \kappa \left[\left(\frac{n\pi}{a} \right)^2 + \left(\frac{m\pi}{b} \right)^2 + \left(\frac{p\pi}{c} \right)^2 \right] = 0$$

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Boundary value problems for heat conduction



Full solution:

$$T(x, y, z, t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \cos\left(\frac{p\pi z}{c}\right) e^{-\lambda_{nmp} t}$$

$$\lambda_{nmp} = \kappa \left[\left(\frac{n\pi}{a} \right)^2 + \left(\frac{m\pi}{b} \right)^2 + \left(\frac{p\pi}{c} \right)^2 \right]$$

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Initial value problem in an infinite domain; Fourier transform

$$\frac{\partial T(\mathbf{r}, t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r}, t) = 0$$

$$T(\mathbf{r}, 0) = f(\mathbf{r})$$

$$\text{Let : } \tilde{T}(\mathbf{q}, t) = \int d^3 r e^{-i\mathbf{q} \cdot \mathbf{r}} T(\mathbf{r}, t)$$

$$\tilde{f}(\mathbf{q}) = \int d^3 r e^{-i\mathbf{q} \cdot \mathbf{r}} f(\mathbf{r})$$

$$\Rightarrow \tilde{T}(\mathbf{q}, 0) = \tilde{f}(\mathbf{q})$$

$$\Rightarrow \frac{\partial \tilde{T}(\mathbf{q}, t)}{\partial t} = -\kappa q^2 \tilde{T}(\mathbf{q}, t)$$

$$\tilde{T}(\mathbf{q}, t) = \tilde{T}(\mathbf{q}, 0) e^{-\kappa q^2 t}$$

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Initial value problem in an infinite domain; Fourier transform

$$\tilde{T}(\mathbf{q}, t) = \int d^3 r e^{-i\mathbf{q} \cdot \mathbf{r}} T(\mathbf{r}, t) \quad \Rightarrow T(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int d^3 q e^{i\mathbf{q} \cdot \mathbf{r}} \tilde{T}(\mathbf{q}, t)$$

$$\tilde{T}(\mathbf{q}, t) = \tilde{T}(\mathbf{q}, 0) e^{-\kappa q^2 t}$$

$$T(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int d^3 q e^{i\mathbf{q} \cdot \mathbf{r}} \tilde{T}(\mathbf{q}, 0) e^{-\kappa q^2 t}$$

$$\tilde{T}(\mathbf{q}, 0) = \tilde{f}(\mathbf{q}) = \int d^3 r e^{-i\mathbf{q} \cdot \mathbf{r}} f(\mathbf{r})$$

$$T(\mathbf{r}, t) = \int d^3 r' G(\mathbf{r} - \mathbf{r}', t) T(\mathbf{r}', 0)$$

$$\text{with } G(\mathbf{r} - \mathbf{r}', t) \equiv \frac{1}{(2\pi)^3} \int d^3 q e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')} e^{-\kappa q^2 t}$$

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Initial value problem in an infinite domain; Fourier transform

$$T(\mathbf{r}, t) = \int d^3 r' G(\mathbf{r} - \mathbf{r}', t) T(\mathbf{r}', 0)$$

$$\text{with } G(\mathbf{r} - \mathbf{r}', t) \equiv \frac{1}{(2\pi)^3} \int d^3 q e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')} e^{-\kappa q^2 t}$$

$$G(\mathbf{r} - \mathbf{r}', t) = \frac{1}{(4\pi\kappa t)^{3/2}} e^{-|\mathbf{r} - \mathbf{r}'|^2 / (4\kappa t)}$$

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Heat equation in half-space

$$\frac{\partial T(\mathbf{r}, t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r}, t) = 0$$

$T(\mathbf{r}, t) \Rightarrow T(z, t)$ with initial and boundary values :

$$T(z, t) \equiv 0 \quad \text{for } z < 0$$

$$T(z, 0) = 0 \quad \text{for } z > 0$$

$$T(0, t) = T_0 \quad \text{for } t \geq 0$$

$$\text{Solution : } T = T_0 \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

$$\text{where } \operatorname{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$$

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Heat equation in half-space -- continued

$$\frac{\partial T(z, t)}{\partial t} - \kappa \frac{\partial^2 T(z, t)}{\partial z^2} = 0$$

$$\text{Solution : } T = T_0 \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

$$\text{where } \operatorname{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$$

$$\text{Note that } \frac{d \operatorname{erfc}(x)}{dx} = \frac{d}{dx} \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du = -\frac{2}{\sqrt{\pi}} e^{-x^2}$$

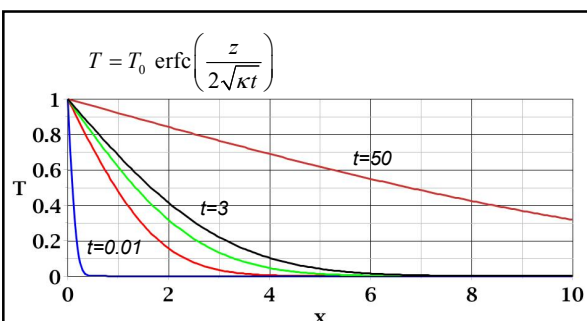
$$\frac{\partial}{\partial t} \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-(z^2/(4\kappa t))} \left(\frac{z}{4\sqrt{\kappa t^3}}\right)$$

$$\frac{\partial^2}{\partial z^2} \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-(z^2/(4\kappa t))} \left(\frac{z}{4\kappa\sqrt{\kappa t^3}}\right)$$

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