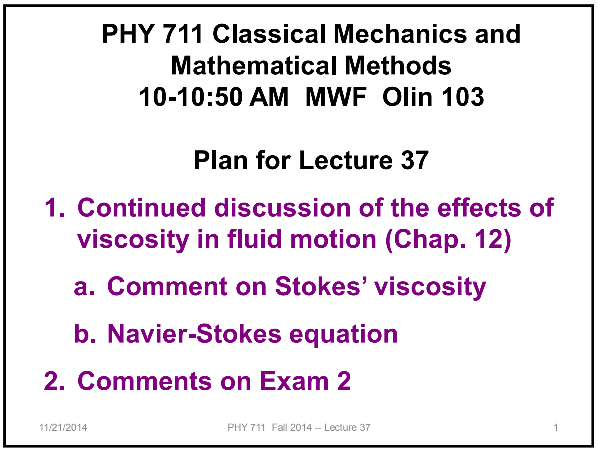
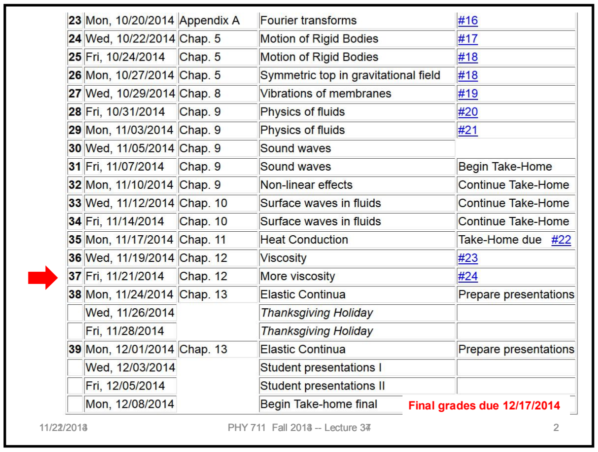


PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

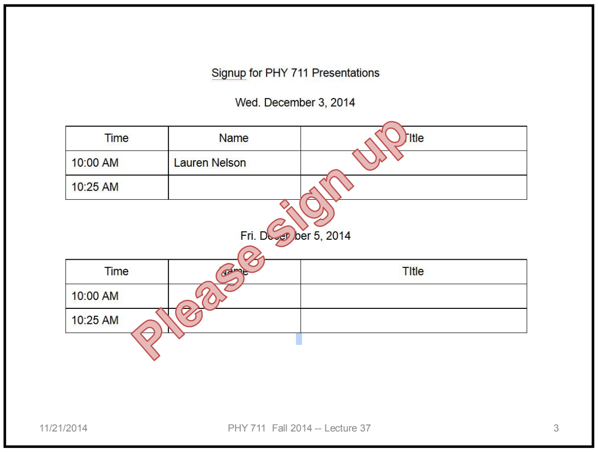
Plan for Lecture 37

1. Continued discussion of the effects of viscosity in fluid motion (Chap. 12)
   a. Comment on Stokes' viscosity
   b. Navier-Stokes equation
2. Comments on Exam 2



23	Mon, 10/20/2014	Appendix A	Fourier transforms	#16
24	Wed, 10/22/2014	Chap. 5	Motion of Rigid Bodies	#17
25	Fri, 10/24/2014	Chap. 5	Motion of Rigid Bodies	#18
26	Mon, 10/27/2014	Chap. 5	Symmetric top in gravitational field	#18
27	Wed, 10/29/2014	Chap. 8	Vibrations of membranes	#19
28	Fri, 10/31/2014	Chap. 9	Physics of fluids	#20
29	Mon, 11/03/2014	Chap. 9	Physics of fluids	#21
30	Wed, 11/05/2014	Chap. 9	Sound waves	
31	Fri, 11/07/2014	Chap. 9	Sound waves	Begin Take-Home
32	Mon, 11/10/2014	Chap. 9	Non-linear effects	Continue Take-Home
33	Wed, 11/12/2014	Chap. 10	Surface waves in fluids	Continue Take-Home
34	Fri, 11/14/2014	Chap. 10	Surface waves in fluids	Continue Take-Home
35	Mon, 11/17/2014	Chap. 11	Heat Conduction	Take-Home due #22
36	Wed, 11/19/2014	Chap. 12	Viscosity	#23
37	Fri, 11/21/2014	Chap. 12	More viscosity	#24
38	Mon, 11/24/2014	Chap. 13	Elastic Continua	Prepare presentations
	Wed, 11/26/2014		*Thanksgiving Holiday*	
	Fri, 11/28/2014		*Thanksgiving Holiday*	
39	Mon, 12/01/2014	Chap. 13	Elastic Continua	Prepare presentations
	Wed, 12/03/2014		Student presentations I	
	Fri, 12/05/2014		Student presentations II	
	Mon, 12/08/2014		Begin Take-home final	

Final grades due 12/17/2014



Signup for PHY 711 Presentations

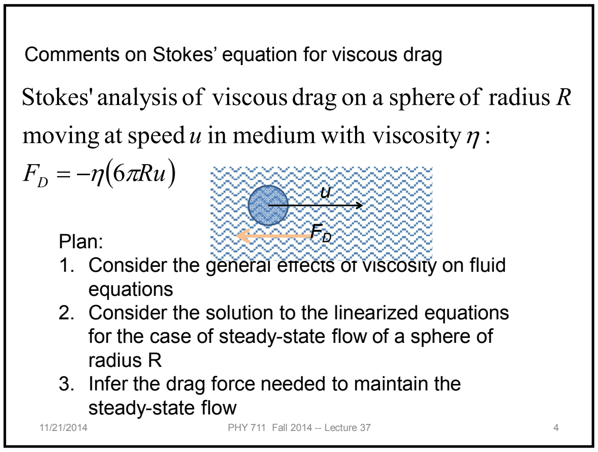
Wed. December 3, 2014

Time	Name	Title
10:00 AM	Lauren Nelson	
10:25 AM		

Fri. December 5, 2014

Time	Name	Title
10:00 AM		
10:25 AM		

Please sign up



Comments on Stokes' equation for viscous drag

Stokes' analysis of viscous drag on a sphere of radius R moving at speed u in medium with viscosity η:

$$F_D = -\eta(6\pi R u)$$

Plan:

1. Consider the general effects of viscosity on fluid equations
2. Consider the solution to the linearized equations for the case of steady-state flow of a sphere of radius R
3. Infer the drag force needed to maintain the steady-state flow

Newton - Euler equation for incompressible fluid, modified by viscous contribution (Navier - Stokes equation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} + \frac{\eta}{\rho}\nabla^2\mathbf{v}$$

Continuity equation: $\nabla\cdot\mathbf{v} = 0$ Irrotational flow: $\nabla\times\mathbf{v} = 0$

Assume steady state: $\Rightarrow \frac{\partial \mathbf{v}}{\partial t} = 0$

Assume non - linear effects small

Initially set $\mathbf{f}_{applied} = 0$; $\quad \nabla p = \eta\nabla^2\mathbf{v}$

Assume $\mathbf{v} = \nabla\times(\nabla\times f(r)\mathbf{u}) + \mathbf{u}$

where $f(r) \xrightarrow[r\to\infty]{} 0$

$$\mathbf{v} = \nabla\times(\nabla\times f(r)\mathbf{u}) + \mathbf{u}$$

$$\mathbf{u} = u\hat{\mathbf{z}}$$

$$\nabla\times(\nabla\times f(r)\hat{\mathbf{z}}) = \nabla(\nabla\cdot f(r)\hat{\mathbf{z}}) - \nabla^2 f(r)\hat{\mathbf{z}}$$

$$\nabla\times\mathbf{v} = 0 \qquad \Rightarrow \nabla^2(\nabla\times\mathbf{v}) = 0$$

$$\nabla^4(\nabla\times f(r)\hat{\mathbf{z}}) = 0 \quad \Rightarrow \nabla^4(\nabla f(r)\times\hat{\mathbf{z}}) = 0 \quad \Rightarrow \nabla^4 f(r) = 0$$

$$f(r) = C_1 r^2 + C_2 r + C_3 + \frac{C_4}{r}$$

$$v_r = u\cos\theta\left(1 - \frac{2}{r}\frac{df}{dr}\right) = u\cos\theta\left(1 - 4C_1 - \frac{2C_2}{r} - \frac{2C_4}{r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1 - \frac{d^2 f}{dr^2} - \frac{1}{r}\frac{df}{dr}\right) = -u\sin\theta\left(1 - 4C_1 - \frac{C_2}{r} + \frac{C_4}{r^3}\right)$$

$$v_r = u\cos\theta\left(1-\frac{2}{r}\frac{df}{dr}\right) = u\cos\theta\left(1-4C_1-\frac{2C_2}{r}-\frac{2C_4}{r^3}\right)$$

$$v_r = u\cos\theta\left(1-\frac{2}{r}\frac{df}{dr}\right) = -u\sin\theta\left(1-4C_1-\frac{C_2}{r}+\frac{C_4}{r^3}\right)$$

To satisfy $\mathbf{v}(r\to\infty)=\mathbf{u}$: $\Rightarrow C_1 = 0$

To satisfy $\mathbf{v}(R)=0$ solve for C_2, C_4

$$v_r = u\cos\theta\left(1-\frac{3R}{2r}+\frac{R^3}{2r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1-\frac{3R}{4r}-\frac{R^3}{4r^3}\right)$$

$$v_r = u\cos\theta\left(1-\frac{3R}{2r}+\frac{R^3}{2r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1-\frac{3R}{4r}-\frac{R^3}{4r^3}\right)$$

Determining pressure:

$$\nabla p = \eta\nabla^2\mathbf{v} = -\eta\nabla\left(u\cos\theta\left(\frac{3R}{2r^2}\right)\right)$$

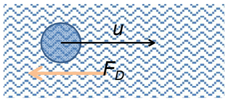
$$\Rightarrow p = p_0 - \eta u\cos\theta\left(\frac{3R}{2r^2}\right)$$

$$p = p_0 - \eta u\cos\theta\left(\frac{3R}{2r^2}\right)$$

Corresponds to:

$$F_D = \left(p(R)-p_0\right)\frac{4\pi R^2}{\cos\theta}$$

$$= -\eta u\left(6\pi R\right)$$



More general formulation of viscosity from Chapter 12 of Fetter and Walecka

For a non-viscous fluid, we can define a stress tensor:

$$T_{kl}^{ideal} = \rho v_k v_l + p\delta_{kl}$$

From Euler and Newton equations for fluid:

$$\int_V d^3r \frac{\partial\left(\rho v_k\right)}{\partial t} = -\sum_{l=1}^{3} \int_A dA_l T_{kl} + \int_V d^3r\, \rho f_k$$

$$-\sum_{l=1}^{3} \int_A dA_l T_{kl} = k\text{th component of force acting on surface } dA$$

For an ideal (non-viscous) fluid: $T_{kl} = T_{lk}$

Effects of viscosity

Argue that viscosity is due to shear forces in a fluid of the form:

$$F_{drag} = \eta \frac{\partial v_x}{\partial y} dA$$

Formulate viscosity stress tensor with traceless and diagonal terms:

$$T_{kl}^{v} = -\eta\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3}\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)\right) - \zeta\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)$$

viscosity

bulk viscosity

Total stress tensor: $T_{kl} = T_{kl}^{ideal} + T_{kl}^{v}$

$$T_{kl}^{ideal} = \rho v_k v_l + p\delta_{kl}$$

$$T_{kl}^{v} = -\eta\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3}\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)\right) - \zeta\delta_{kl}\left(\nabla\cdot\mathbf{v}\right)$$

Effects of viscosity -- continued

Incorporating generalized stress tensor into Newton-Euler equations

$$\frac{\partial \rho v_i}{\partial t} + \sum_{j=1}^{3}\frac{\partial \rho v_i v_j}{\partial x_j} = \rho f_i - \frac{\partial p}{\partial x_i} + \eta\sum_{j=1}^{3}\frac{\partial^2 v_i}{\partial x_j^2} + \left(\zeta + \frac{1}{3}\eta\right)\sum_{j=1}^{3}\frac{\partial^2 v_j}{\partial x_i \partial x_j}$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3}\frac{\partial \rho v_j}{\partial x_j} = 0$$

Vector form (Navier-Stokes equation)

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}\cdot\nabla\right)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla\left(\nabla\cdot\mathbf{v}\right)$$

Comments on Exam 2

1. Consider a displacement function $u(x,t)$ representing a one-dimensional traveling wave (either transverse or longitudinal) which is a solution of the one-dimensional wave equation with wave speed c:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0.$$

If the initial conditions for the wave displacement $u(x,t)$ are given by

$$u(x,0) = U_0 \mathrm{e}^{-(x-x_0)^2/\sigma^2},$$

and

$$\frac{\partial u}{\partial t}(x,0) = V_0 \left(\frac{x}{\mu}\right)^3 \mathrm{e}^{-(x/\mu)^4},$$

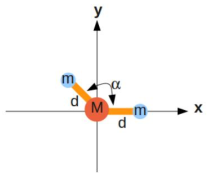
find the form of $u(x,t)$ for $t > 0$. Express your result in terms of the constants U_0, V_0, σ, μ, x_0, and c.

Solution for $u(x,0) \equiv \varphi(x)$ and $\dfrac{\partial u}{\partial t}(x,0) \equiv \psi(x)$:

$$\Rightarrow u(x,t) = \frac{1}{2}\big(\varphi(x-ct) + \varphi(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x')dx'$$

$$u(x,t) = \frac{U_0}{2}\left(e^{-(x-x_0+ct)/\sigma^2} + e^{-(x-x_0-ct)/\sigma^2}\right)$$

$$+\frac{V_0\mu}{8c}\left(e^{-((x+ct)/\mu)^4} - e^{-((x-ct)/\mu)^4}\right)$$



2.

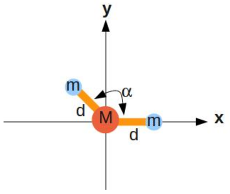
The figure above shows a triatomic molecule aligned as shown with $M = 16m$ at the origin and the other two masses m in the $x-y$ plane at the locations $(d\hat{\mathbf{x}})$ and $(d\cos(\alpha)\hat{\mathbf{x}} + d\sin(\alpha)\hat{\mathbf{y}})$.

(a) Find the moment of inertial tensor for the molecule in the given coordinate system, expressing your answer in terms of the parameters m, d, and α.

(b) Find the principal moments of inertia and the corresponding principal axes for this coordinate system.

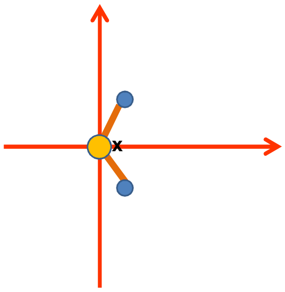
(c) Find the center of mass of this system.

(d) Find the principal moments of inertia and corresponding principal axes for rotating this system about its center of mass.



$$I = md^2 \begin{pmatrix} \sin^2\alpha & -\sin\alpha\cos\alpha & 0 \\ -\sin\alpha\cos\alpha & 1+\cos^2\alpha & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Principal moments of inertia



. The 3 Euler angles, α, β, and γ were defined as a series of rotations to take a vector $\mathbf{v}$ in one coordinate frame to another frame in which is written as $\mathbf{v}'$. Alternatively, these angles can represent a general rotation of a vector quantity within a given coordinate frame.

$$\mathbf{v}' = \mathcal{R}\mathbf{v} \quad \text{where} \quad \mathcal{R} = \mathcal{R}_\gamma\mathcal{R}_\beta\mathcal{R}_\alpha.$$

It can be shown that the 3×3 matrix components of the rotation matrix are given by:

$$\mathcal{R}_{xx} = \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma$$
$$\mathcal{R}_{xy} = \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma$$
$$\mathcal{R}_{xz} = -\sin\beta\cos\gamma$$
$$\mathcal{R}_{yx} = -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma$$
$$\mathcal{R}_{yy} = -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma$$
$$\mathcal{R}_{yz} = \sin\beta\sin\gamma$$
$$\mathcal{R}_{zx} = \cos\alpha\sin\beta$$
$$\mathcal{R}_{zy} = \sin\alpha\sin\beta$$
$$\mathcal{R}_{zz} = \cos\beta$$

As an example, consider the vector

$$\mathbf{v} = \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix}$$

(a) Evaluate the matrix $\mathcal{R}$ and find the vector $\mathbf{v}' = \mathcal{R}\mathbf{v}$ when $\alpha = \pi/4$, $\cos(\beta) = \sqrt{1/3}$, and $\gamma = -\pi/3$.

(b) Evaluate the matrix $\mathcal{R}$ and find the vector $\mathbf{v}' = \mathcal{R}\mathbf{v}$ when $\alpha = \pi/4$, $\cos(\beta) = \sqrt{1/3}$, and $\gamma = \pi/3$.

a) $\mathcal{R} = \begin{pmatrix} \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{6}} \\ 0 & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix}$ $\quad \mathbf{v}' = \mathcal{R}\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

b) $\mathcal{R} = \begin{pmatrix} -\sqrt{\frac{1}{6}} & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{6}} \\ -\sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix}$ $\quad \mathbf{v}' = \mathcal{R}\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Consider linear sound waves in a gas confined between two rigid concentric spheres of radii a and $3a$. The velocity potential $\Phi(\mathbf{r},t)$ for the sound wave satisfies the equation

$$\frac{\partial^2\Phi}{\partial t^2} - c^2\nabla^2\Phi = 0,$$

where c denotes the speed of sound (assumed to be constant). Because of the spherical geometry, it is convenient to factor the velocity potential into radial, angular, and time dependent terms:

$$\Phi(r,\theta,\phi,t) = f_l(r)Y_{lm}(\theta,\phi)\mathrm{e}^{-i\omega t},$$

where ω denotes a harmonic frequency and $Y_{lm}(\theta,\phi)$ denotes the angular function which satisfies the equation

$$\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)Y_{lm}(\theta,\phi) = -l(l+1)Y_{lm}(\theta,\phi).$$

(a) Find the equation that the radial functions $f_l(r)$ must satisfy to represent the radial part of the normal modes of the sound waves.

(b) Show that the boundary conditions for the normal mode functions are

$$\frac{df_l(a)}{dr} = 0 \quad \text{and} \quad \frac{df_l(3a)}{dr} = 0.$$

(c) Numerically determine the lowest frequency spherically symmetric ($l = 0$) normal mode. Find frequency ω as a multiple of the speed of sound c and the corresponding radial function.

(d) Numerically determine the lowest frequency $l = 1$ normal mode. Find frequency ω as a multiple of the speed of sound c and the corresponding radial function.

Note – the following functional forms for the spherical Bessel functions may prove useful:

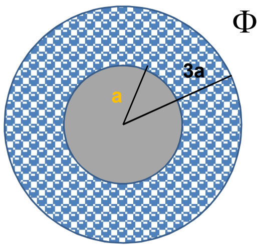
$$j_0(x) = \frac{\sin(x)}{x} \qquad y_0(x) = -\frac{\cos(x)}{x}$$

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \qquad y_1(x) = -\frac{\cos(x)}{x^2} - \frac{\sin(x)}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin(x) - \frac{3\cos(x)}{x^2} \qquad y_2(x) = \left(-\frac{3}{x^3} + \frac{1}{x}\right)\cos(x) - \frac{3\sin(x)}{x^2}$$

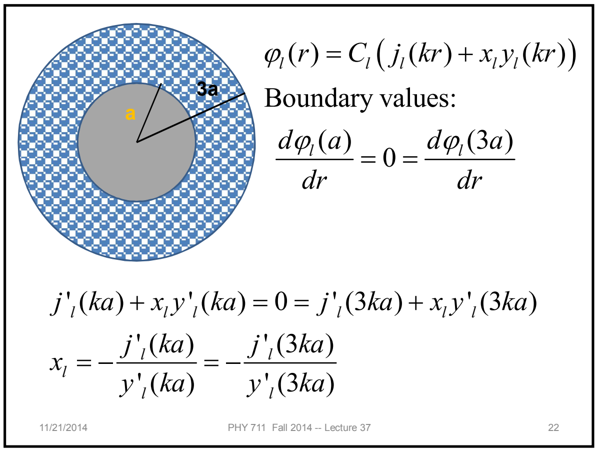
These spherical Bessel and Neumann functions ($z_l(x)$ for $z = j$ or $z = y$) satisfy the differential equation:

$$\left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{l(l+1)}{x^2} + 1\right)z_l(x) = 0.$$



$$\Phi(\mathbf{r},t) = \phi_l(r)Y_{lm}(\hat{\mathbf{r}})e^{-i\omega t}$$

$$\phi_l(r) = C_l\left(j_l(kr) + x_l y_l(kr)\right)$$



$$\varphi_l(r) = C_l\left(j_l(kr) + x_l y_l(kr)\right)$$

Boundary values:

$$\frac{d\varphi_l(a)}{dr} = 0 = \frac{d\varphi_l(3a)}{dr}$$

$$j'_l(ka) + x_l y'_l(ka) = 0 = j'_l(3ka) + x_l y'_l(3ka)$$

$$x_l = -\frac{j'_l(ka)}{y'_l(ka)} = -\frac{j'_l(3ka)}{y'_l(3ka)}$$

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