PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 38

1. **Chapter 12: Example of solution to Navier-Stokes equation**
2. **Chapter 13: Physics of elastic continua**

23	Mon, 10/20/2014	Appendix A	Fourier transforms	#16
24	Wed, 10/22/2014	Chap. 5	Motion of Rigid Bodies	#17
25	Fri, 10/24/2014	Chap. 5	Motion of Rigid Bodies	#18
26	Mon, 10/27/2014	Chap. 5	Symmetric top in gravitational field	#18
27	Wed, 10/29/2014	Chap. 8	Vibrations of membranes	#19
28	Fri, 10/31/2014	Chap. 9	Physics of fluids	#20
29	Mon, 11/03/2014	Chap. 9	Physics of fluids	#21
30	Wed, 11/05/2014	Chap. 9	Sound waves	
31	Fri, 11/07/2014	Chap. 9	Sound waves	Begin Take-Home
32	Mon, 11/10/2014	Chap. 9	Non-linear effects	Continue Take-Home
33	Wed, 11/12/2014	Chap. 10	Surface waves in fluids	Continue Take-Home
34	Fri, 11/14/2014	Chap. 10	Surface waves in fluids	Continue Take-Home
35	Mon, 11/17/2014	Chap. 11	Heat Conduction	Take-Home due #22
36	Wed, 11/19/2014	Chap. 12	Viscosity	#23
37	Fri, 11/21/2014	Chap. 12	More viscosity	#24
38	Mon, 11/24/2014	Chap. 13	Elastic Continua	Prepare presentations
	Wed, 11/26/2014		*Thanksgiving Holiday*	
	Fri, 11/28/2014		*Thanksgiving Holiday*	
39	Mon, 12/01/2014	Chap. 13	Elastic Continua	Prepare presentations
	Wed, 12/03/2014		Student presentations I	
	Fri, 12/05/2014		Student presentations II	
	Mon, 12/08/2014		Begin Take-home final	Final grades due 12/17/2014

Signup for PHY 711 Presentations

Wed. December 3, 2014

Time	Name	Title
10:00 AM	Lauren Nelson	
10:25 AM		

Fri. December 5, 2014

Time	Name	Title
10:00 AM		
10:25 AM		

Please sign up

Newton-Euler equations for viscous fluids

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R*

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Incompressible fluid $\Rightarrow \nabla\cdot\mathbf{v} = 0$

Steady flow $\Rightarrow \frac{\partial \mathbf{v}}{\partial t} = 0$

Irrotational flow $\Rightarrow \nabla\times\mathbf{v} = 0$

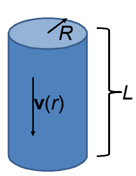
No applied force $\Rightarrow \mathbf{f}=0$

Neglect non-linear terms $\Rightarrow \nabla\left(v^2\right) = 0$

Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R* -- continued

Navier-Stokes equation becomes:

$$0 = -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v}$$



Assume that $\mathbf{v}(\mathbf{r},t) = v_z(r)\hat{\mathbf{z}}$

$$\frac{\partial p}{\partial z} = \eta\nabla^2 v_z(r) \quad \text{(independent of } z\text{)}$$

Suppose that $\frac{\partial p}{\partial z} = -\frac{\Delta p}{L}$

$$\Rightarrow \nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius R -- continued

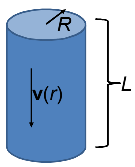
$$\nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

$$\frac{1}{r}\frac{d}{dr} r \frac{dv_z(r)}{dr} = -\frac{\Delta p}{\eta L}$$

$$v_z(r) = -\frac{\Delta p r^2}{4\eta L} + C_1 \ln(r) + C_2$$

$$\Rightarrow C_1 = 0 \qquad v_z(R) = 0 = -\frac{\Delta p R^2}{4\eta L} + C_2$$

$$v_z(r) = \frac{\Delta p}{4\eta L}\left(R^2 - r^2\right)$$

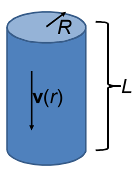


Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius R -- continued

$$v_z(r) = \frac{\Delta p}{4\eta L}\left(R^2 - r^2\right)$$

Mass flow rate through the pipe:

$$\frac{dM}{dt} = 2\pi\rho \int_0^R r dr v_z(r) = \frac{\Delta p \rho \pi R^4}{8\eta L}$$



Poiseuille formula;
→Method for measuring η

Newton-Euler equations for viscous fluids – effects on sound

Recall full equations:

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Newton-Euler equations for viscous fluids – effects on sound

Without viscosity terms:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p \qquad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Assume: $\mathbf{v} = 0 + \delta\mathbf{v} \qquad \mathbf{f}=0 \qquad \rho = \rho_0 + \delta\rho$

$$p = p_0 + \delta p = p_0 + c^2\delta\rho$$

Linearized equations: $\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla\delta\rho \qquad \frac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot(\delta\mathbf{v}) = 0$

Let $\delta\mathbf{v} \equiv \delta\mathbf{v}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \delta\rho \equiv \delta\rho_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

$$\Rightarrow \omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} \qquad -\omega\delta\rho_0 + \rho_0\mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\Rightarrow k^2 = \frac{\omega^2}{c^2} \qquad \frac{\delta\rho_0}{\rho_0} = \frac{\hat{\mathbf{k}}\cdot\delta\mathbf{v}_0}{c}$$

➔Pure longitudinal harmonic wave solutions

Newton-Euler equations for viscous fluids – effects on sound

Recall full equations:

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Assume: $\mathbf{v} = 0 + \delta\mathbf{v} \qquad \mathbf{f}=0 \qquad \rho = \rho_0 + \delta\rho$

$$p = p_0 + \delta p = p_0 + c^2\delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \delta s$$

where $c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s$

Newton-Euler equations for viscous fluids – effects on sound

Note that pressure now depends both on density and entropy so that entropy must be coupled into the equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \qquad \rho T\frac{\partial s}{\partial t} = k_{th}\nabla^2 T$$

Assume: $\mathbf{v} = 0 + \delta\mathbf{v} \qquad \mathbf{f}=0 \qquad \rho = \rho_0 + \delta\rho$

$$p = p_0 + \delta p = p_0 + c^2\delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \delta s \quad \text{where } c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s$$

$$T = T_0 + \delta T = T_0 + \left(\frac{\partial T}{\partial \rho}\right)_s \delta\rho + \left(\frac{\partial T}{\partial s}\right)_\rho \delta s$$

$$s = s_0 + \delta s$$

Newton-Euler equations for viscous fluids – effects on sound
Linearized equations (with the help of various thermodynamic relationships):

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla \delta\rho - \rho_0\left(\frac{\partial T}{\partial \rho}\right)_s \nabla \delta s + \frac{\eta}{\rho_0}\nabla^2 \delta \mathbf{v} + \frac{1}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\nabla\left(\nabla \cdot \delta \mathbf{v}\right)$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \left(\delta \mathbf{v}\right) = 0$$

$$\frac{\partial \delta s}{\partial t} = \gamma\kappa \nabla^2 \delta s + \frac{c_p \kappa}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \nabla^2 \delta\rho$$

Here: $\gamma = \frac{c_p}{c_v}$ $\qquad \kappa = \frac{k_{th}}{c_p \rho_0}$

Let $\delta \mathbf{v} \equiv \delta \mathbf{v}_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ $\quad \delta\rho \equiv \delta\rho_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ $\quad \delta s \equiv \delta s_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$

Newton-Euler equations for viscous fluids – effects on sound
Linearized plane wave solutions:

$$\omega \delta \mathbf{v}_0 = -\frac{c^2 \delta\rho_0}{\rho_0}\mathbf{k} - \rho_0\left(\frac{\partial T}{\partial \rho}\right)_s \delta s_0 \mathbf{k} - \frac{i\eta k^2}{\rho_0}\delta \mathbf{v}_0 - \frac{i}{\rho_0}\left(\zeta + \frac{1}{3}\eta\right)\mathbf{k}\left(\mathbf{k}\cdot \delta \mathbf{v}_0\right)$$

$$\frac{\partial \delta\rho_0}{\partial t} - \rho_0 \; \mathbf{k}\cdot \delta \mathbf{v}_0 = 0$$

$$\omega \delta s_0 = -i\gamma\kappa k^2 \delta s_0 - \frac{i c_p \kappa}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s k^2 \delta\rho_0$$

Longitudinal solutions:

$$\left(\omega^2 - c^2 k^2 + i\frac{\omega k^2}{\rho_0}\left(\frac{4}{3}\eta + \zeta\right)\right)\delta\rho_0 - \rho_0^2 k^2 \left(\frac{\partial T}{\partial \rho}\right)_s \delta s_0 = 0$$

$$\frac{i c_p \kappa k^2}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \delta\rho_0 + \left(\omega + i\gamma\kappa k^2\right)\delta s_0 = 0$$

Newton-Euler equations for viscous fluids – effects on sound
Linearized plane wave longitudinal solutions:

Longitudinal solutions:

$$\left(\omega^2 - c^2 k^2 + i\frac{\omega k^2}{\rho_0}\left(\frac{4}{3}\eta + \zeta\right)\right)\delta\rho_0 - \rho_0^2 k^2 \left(\frac{\partial T}{\partial \rho}\right)_s \delta s_0 = 0$$

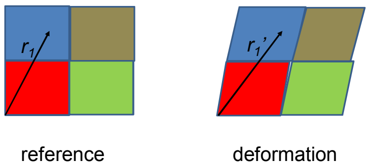
$$\frac{i c_p \kappa k^2}{T_0}\left(\frac{\partial T}{\partial \rho}\right)_s \delta\rho_0 + \left(\omega + i\gamma\kappa k^2\right)\delta s_0 = 0$$

Approximate solution: $\quad k = \frac{\omega}{c} + i\alpha$

where $\quad \alpha \approx \frac{\omega^2}{2c^3 \rho_0}\left(\frac{4}{3}\eta + \zeta\right) + \frac{\kappa c_p \rho_0^2 \omega^2}{2T_0 c^5}\left(\frac{\partial T}{\partial \rho}\right)_s^2$

$$\delta\rho = \delta\rho_0 e^{-\alpha \hat{\mathbf{k}}\cdot\mathbf{r}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

Brief introduction to elastic continua



$$\mathbf{r}_1' = \mathbf{r}_1 + \mathbf{u}(\mathbf{r}_1) \qquad \mathbf{r}_2' = \mathbf{r}_2 + \mathbf{u}(\mathbf{r}_2)$$

$$\mathbf{r}_2' - \mathbf{r}_1' = \mathbf{r}_2 - \mathbf{r}_1 + \left((\mathbf{r}_2 - \mathbf{r}_1)\cdot\nabla\right)\mathbf{u}(\mathbf{r}_1)$$

Brief introduction to elastic continua

Deformation components:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\equiv \epsilon_{ij} + O_{ij}$$

To be continued ….

Have a great Thanksgiving Holiday.