PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 39

1. Brief introduction to the physics of elastic continua (Chap. 13 of F & W)
2. Brief review of topics covered this semester
3. Course evaluation forms

23	Mon, 10/20/2014	Appendix A	Fourier transforms	#16
24	Wed, 10/22/2014	Chap. 5	Motion of Rigid Bodies	#17
25	Fri, 10/24/2014	Chap. 5	Motion of Rigid Bodies	#18
26	Mon, 10/27/2014	Chap. 5	Symmetric top in gravitational field	#18
27	Wed, 10/29/2014	Chap. 8	Vibrations of membranes	#19
28	Fri, 10/31/2014	Chap. 9	Physics of fluids	#20
29	Mon, 11/03/2014	Chap. 9	Physics of fluids	#21
30	Wed, 11/05/2014	Chap. 9	Sound waves	
31	Fri, 11/07/2014	Chap. 9	Sound waves	Begin Take-Home
32	Mon, 11/10/2014	Chap. 9	Non-linear effects	Continue Take-Home
33	Wed, 11/12/2014	Chap. 10	Surface waves in fluids	Continue Take-Home
34	Fri, 11/14/2014	Chap. 10	Surface waves in fluids	Continue Take-Home
35	Mon, 11/17/2014	Chap. 11	Heat Conduction	Take-Home due #22
36	Wed, 11/19/2014	Chap. 12	Viscosity	#23
37	Fri, 11/21/2014	Chap. 12	More viscosity	#24
38	Mon, 11/24/2014	Chap. 13	Elastic Continua	Prepare presentations
	Wed, 11/26/2014		*Thanksgiving Holiday*	
	Fri, 11/28/2014		*Thanksgiving Holiday*	
39	Mon, 12/01/2014	Chap. 13	Elastic Continua	Prepare presentations
	Wed, 12/03/2014		Student presentations I	
	Fri, 12/05/2014		Student presentations II	
	Mon, 12/08/2014		Begin Take-home final	**Final grades due 12/17/2014**

Signup for PHY 711 Presentations

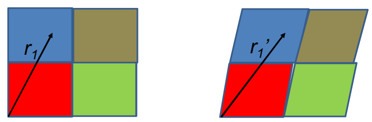
Wed. December 3, 2014

Time	Name	Title
10:00 AM	Lauren Nelson	Molecular Dynamics Simulations in VMD and NAMD Programs (working title)
10:25 AM	Larry Rush	Bessel Function's utilty for Digital Processing (Kaiser Window)

Fri. December 5, 2014

Time	Name	Title
10:00 AM	Ritchie Dudley	Sound of the Conch Shell: It's all in your head
10:25 AM	Jason Howard	Simple model of swimming

Brief introduction to elastic continua



reference   deformation

$$\mathbf{r}_1' = \mathbf{r}_1 + \mathbf{u}(\mathbf{r}_1) \qquad \mathbf{r}_2' = \mathbf{r}_2 + \mathbf{u}(\mathbf{r}_2)$$

$$\mathbf{r}_2' - \mathbf{r}_1' = \mathbf{r}_2 - \mathbf{r}_1 + \left((\mathbf{r}_2 - \mathbf{r}_1)\cdot\nabla\right)\mathbf{u}(\mathbf{r}_1)$$

Brief introduction to elastic continua

Deformation components:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\equiv \epsilon_{ij} + O_{ij}$$

$$V = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \qquad V' = \mathbf{a}'\cdot(\mathbf{b}'\times\mathbf{c}') \qquad V' = V\left(1 + \mathrm{Tr}(\epsilon)\right) = V(1 + \nabla\cdot\mathbf{u})$$

$$\mathrm{Tr}(\epsilon) = \nabla\cdot\mathbf{u} = \frac{dV}{V} = -\frac{d\rho}{\rho}$$

Stress tensor

$-\sum_{j=1}^{3} T_{ij} dA_j \Rightarrow i^{th}$ component of force acting on surface $\hat{\mathbf{n}}$

Generalization of Hooke's law, $F_x = -kx$:

Lame' coefficients: $T_{ij} = -\lambda\delta_{ij}\nabla\cdot\mathbf{u} - \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

or: $T_{ij} = -\lambda\delta_{ij}\mathrm{Tr}(\epsilon) - 2\mu\epsilon_{ij}$

Note that: $\mathrm{Tr}(T) = -3\underbrace{\left(\lambda + \frac{2}{3}\mu\right)}_{K}\mathrm{Tr}(\epsilon)$

$K \equiv$ bulk modulus $= -V\left(\frac{\partial p}{\partial V}\right)$

$$\Rightarrow \epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)}\delta_{ij} Tr(T)\right)$$

$$\epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)}\,\delta_{ij}Tr(T)\right)$$

Example -- hydrostatic pressure: $T_{ij} = \delta_{ij}dp$

$$\epsilon_{ij} = -\frac{dp}{3\left(\lambda + \frac{2}{3}\mu\right)}\delta_{ij} \equiv -\frac{dp}{3K}\delta_{ij}$$

$$\Rightarrow K = -V\frac{\partial p}{\partial V}$$

Example -- uniaxial pressure: $T_{ij} = \begin{cases} dp & ij = zz \\ 0 & \text{otherwise} \end{cases}$

$$\epsilon_{zz} = -\frac{1}{E}T_{zz} \quad \text{in terms of Young's modulus}$$

$$E = \frac{9K\mu}{3K + \mu}$$

Dynamical equations of elastic continuum

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu\nabla^2\mathbf{u} + \left(K + \frac{1}{3}\mu\right)\nabla\left(\nabla\cdot\mathbf{u}\right) + \rho\mathbf{f}$$

In the absence of external forces, this reduces to two decoupled wave equations representing longitudinal and transverse modes:

$\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$

where $\nabla\times\mathbf{u}_l = 0$ and $\nabla\cdot\mathbf{u}_t = 0$

$$c_l = \left(\frac{K + \frac{4}{3}\mu}{\rho}\right)^{1/2} \quad \text{and} \quad c_t = \left(\frac{\mu}{\rho}\right)^{1/2}$$

Brief review of topics covered in this course --
Scattering theory:

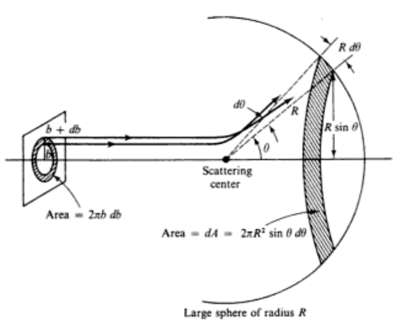


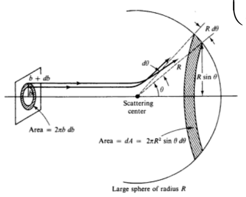
Figure 5.5 The scattering problem and relation of cross section to impact parameter.

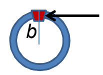
Differential cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

= Area of incident beam that is scattered into detector at angle θ

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{d\phi\, b\, db}{d\phi\, \sin\theta\, d\theta} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$





Scattering angle equation for central potential $V(r)$:

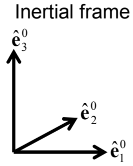
$$\theta = \pi - 2b \int_0^{1/r_{min}} du \left(\frac{1}{\sqrt{1 - b^2u^2 - \frac{V(1/u)}{E}}} \right)$$

Rotating reference frames

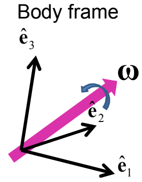
$$\left(\frac{d}{dt}\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega}\times\right)\left\{\left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{V}\right\}$$

$$\left(\frac{d^2\mathbf{V}}{dt^2}\right)_{inertial} = \left(\frac{d^2\mathbf{V}}{dt^2}\right)_{body} + 2\boldsymbol{\omega}\times\left(\frac{d\mathbf{V}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{V} + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{V}$$

Inertial frame



Body frame



Newton's laws; Let **r** denote the position of particle of mass *m*:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{inertial} = \mathbf{F}_{ext}$$

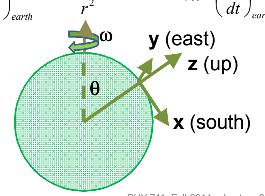
$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{body} = \mathbf{F}_{ext} - m\left(\frac{d^2\mathbf{a}}{dt^2}\right)_{inertial} - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{body} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

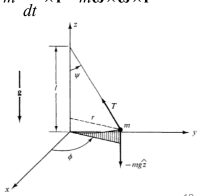
Coriolis force

Centrifugal force

Equation of motion on Earth's surface

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$





Calculus of variation example for a pure integral functions

Find the function $y(x)$ which extremizes $I\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$

where $I\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) \equiv \int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx.$

Necessary condition: $\delta I = 0$

Euler-Lagrange equations:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

Application to particle dynamics

$x \to t$ (time)

$y \to q$ (generalized coordinate)

$f \to L$ (Lagrangian)

$I \to S$ (action)

Denote: $\dot{q} \equiv \frac{dq}{dt}$

$$S = \int_{t_1}^{t_2} L\left(\{q,\dot{q}\};t\right)dt$$

Euler-Lagrange equations:

$$\left(\frac{\partial L}{\partial q}\right)_{t,\frac{dq}{dt}} - \frac{d}{dt}\left[\left(\frac{\partial L}{\partial(dq/dt)}\right)_{t,q}\right] = 0$$

The Lagrangian is given by:

$$L\left(\left\{\mathbf{r}(t),\frac{d\mathbf{r}}{dt}\right\},t\right) \equiv T - U$$

T: Kinetic energy; U: Potential energy

Note: For a particle of charge q, the potential function U can have the form

$$U = U_0(\mathbf{r},t) + q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$$

where $U_0(\mathbf{r},t)$ denotes non-electromagnetic field contributions and

where $\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}$ $\qquad \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t)$

Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function : $L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$
2. Compute generalized momenta : $p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$
3. Construct Hamiltonian expression : $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$
4. Form Hamiltonian function : $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$
5. Analyze canonical equations of motion :

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

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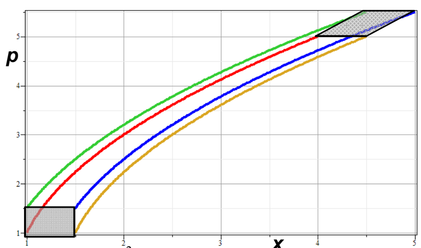
Notion of phase space – example for one-dimensional motion due to constant force



$$H(x,p) = \frac{p^2}{2m} - F_0 x \qquad \dot{p} = F_0 \qquad \dot{x} = \frac{p}{m}$$

$$p_i(t) = p_{0i} + F_0 t \qquad x_i(t) = x_{0i} + \frac{p_{0i}}{m} t + \frac{1}{2} F_0 t^2$$

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Liouville's Theorem (1838)

The density of representative points in phase space corresponding to the motion of a system of particles remains constant during the motion.

Denote the density of particles in phase space : $D = D(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$\frac{dD}{dt} = \sum_\sigma \left(\frac{\partial D}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial D}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial D}{\partial t}$$

According to Liouville's theorem : $\frac{dD}{dt} = 0$

Importance of Liouville's theorem to statistical mechanical analysis: In statistical mechanics, we need to evaluate the probability of various configurations of particles. The fact that the density of particles in phase space is constant in time, implies that each point in phase space is equally probable and that the time average of the evolution of a system can be determined by an average of the system over phase space volume.

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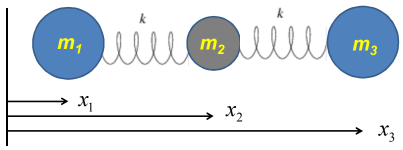
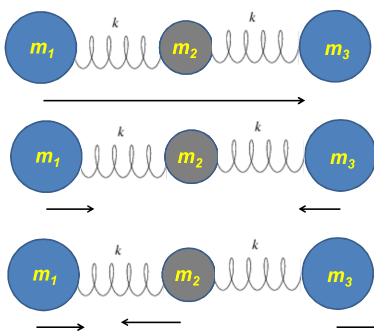
Analysis of small oscillations near equilibrium

Example – linear molecule



$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2$$
$$-\frac{1}{2}k\left(x_2 - x_1 - \ell_{12}\right)^2 - \frac{1}{2}k\left(x_3 - x_2 - \ell_{23}\right)^2$$

Normal modes of oscillation for linear molecule example--

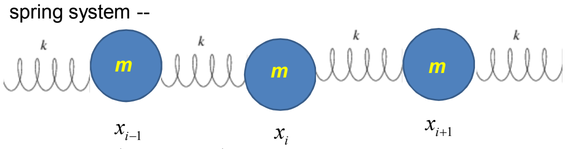


$$\omega_1 = 0$$

$$\omega_2 = \sqrt{\frac{k}{m_O}}$$

$$\omega_3 = \sqrt{\frac{k}{m_O} + \frac{2k}{m_C}}$$

Longitudinal waves as the continuum limit of a linear mass-spring system --



$$L = T - V = \frac{1}{2}m\sum_{i=0}^{\infty}\dot{x}_i^2 - \frac{1}{2}k\sum_{i=0}^{\infty}\left(x_{i+1} - x_i\right)^2$$

$$\Rightarrow m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1}\right)$$

Now imagine the continuum version of this system:

$$x_i(t) \Rightarrow \mu(x_i, t) \qquad \ddot{x}_i \Rightarrow \frac{\partial^2\mu}{\partial t^2}$$

$$x_{i+1} - 2x_i + x_{i-1} \Rightarrow \frac{\partial^2\mu}{\partial x^2}(\Delta x)^2$$

Discrete equation : $m\ddot{x}_i = k\left(x_{i+1} - 2x_i + x_{i-1}\right)$

Continuum equation : $m\dfrac{\partial^2 \mu}{\partial t^2} = k(\Delta x)^2 \dfrac{\partial^2 \mu}{\partial x^2}$

$$\frac{\partial^2 \mu}{\partial t^2} = \left(\frac{k\Delta x}{m/\Delta x}\right)\frac{\partial^2 \mu}{\partial x^2}$$

system parameter with units of (velocity)^2=c^2

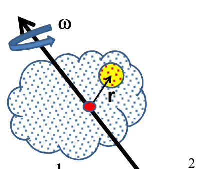
Wave equation for longitudinal or transverse displacement $\mu(x,t)$:

$$\frac{\partial^2 \mu}{\partial t^2} - c^2 \frac{\partial^2 \mu}{\partial x^2} = 0$$

Solution methods for wave equations and generalizations

1. D'Alembert's method
2. Fourier transforms
3. Laplace transforms; contour integration
4. Eigenfunction expansions
5. Green's functions
6. Variational methods

Physics of rigid bodies rotating about a fixed origin



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

Kinetic energy: $T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\left|\boldsymbol{\omega} \times \mathbf{r}_p\right|\right)^2$

$$= \sum_p \frac{1}{2} m_p \left(\boldsymbol{\omega} \times \mathbf{r}_p\right) \cdot \left(\boldsymbol{\omega} \times \mathbf{r}_p\right)$$

$$= \sum_p \frac{1}{2} m_p \left[\left(\boldsymbol{\omega} \cdot \boldsymbol{\omega}\right)\left(\mathbf{r}_p \cdot \mathbf{r}_p\right) - \left(\mathbf{r}_p \cdot \boldsymbol{\omega}\right)^2\right]$$

$$= \boldsymbol{\omega} \cdot \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega}$$

Moment of inertia tensor

Matrix notation:

$$\ddot{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \qquad I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj} \right)$$

For general coordinate system: $T = \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor: $\ddot{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad i = 1,2,3$

$$\boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3 \qquad \Rightarrow T = \frac{1}{2}\sum_i I_i \tilde{\omega}_i^2$$

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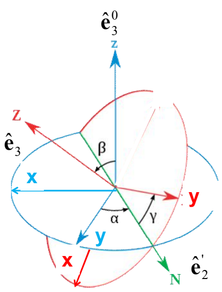
Euler angles for describing angular velocity ω in terms of body fixed contributions --

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2' + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



$$\begin{aligned} \tilde{\boldsymbol{\omega}} = & \left[\dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma\right]\hat{\mathbf{e}}_1 \\ & + \left[\dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma\right]\hat{\mathbf{e}}_2 \\ & + \left[\dot{\alpha}\cos\beta + \dot{\gamma}\right]\hat{\mathbf{e}}_3 \end{aligned}$$

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Equations describing fluid physics

Newton-Euler equation of motion:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity condition: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

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Solution of equations in the linear approximation – (linear sound waves)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p \qquad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Assume: $\mathbf{v} = 0 + \delta\mathbf{v}$ $\quad \mathbf{f}=0 \quad$ $\rho = \rho_0 + \delta\rho$

$$p = p_0 + \delta p = p_0 + c^2\delta\rho$$

Linearized equations: $\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{c^2}{\rho_0}\nabla\delta\rho \qquad \frac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot(\delta\mathbf{v}) = 0$

Let $\delta\mathbf{v} \equiv \delta\mathbf{v}_0\; e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \delta\rho \equiv \delta\rho_0\; e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

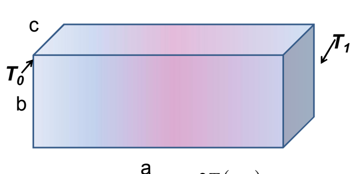
$$\Rightarrow \omega\delta\mathbf{v}_0 = \frac{c^2\delta\rho_0}{\rho_0}\mathbf{k} \qquad -\omega\delta\rho_0 + \rho_0\mathbf{k}\cdot\delta\mathbf{v}_0 = 0$$

$$\Rightarrow k^2 = \frac{\omega^2}{c^2} \qquad \frac{\delta\rho_0}{\rho_0} = \frac{\hat{\mathbf{k}}\cdot\delta\mathbf{v}_0}{c}$$

Wave analysis in fluid physics beyond linear regime

1. Non-linear effects in sound waves – shock phenomena
2. Application of Euler-Newton equations to surface waves in water
3. Korteweg-de Vries soliton analysis

Analysis of heat conduction



$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = \frac{\dot{q}}{c_p}$$

$$\kappa = \frac{k_{th}}{\rho c_p}$$

Newton-Euler equations for viscous fluids

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2\mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Presentations –

After your presentation, please send me your slides and notes. In order to encourage discussion after the talks, points will be awarded for peer questions.

Wed. December 3, 2014

Time	Name	Title
10:00 AM	Lauren Nelson	Molecular Dynamics Simulations in VMD and NAMD Programs (working title)
10:25 AM	Larry Rush	Bessel Function's utilty for Digital Processing (Kaiser Window)

Fri. December 5, 2014

Time	Name	Title
10:00 AM	Ritchie Dudley	Sound of the Conch Shell: It's all in your head
10:25 AM	Jason Howard	Simple model of swimming