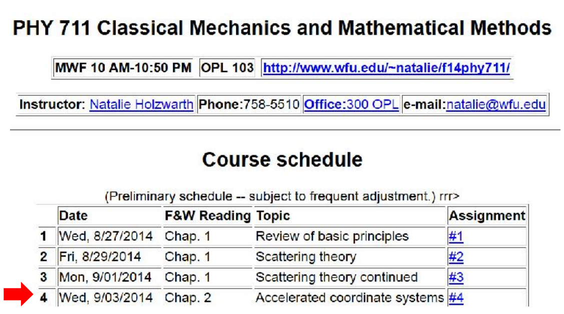
PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 4:

Chapter 2 – Physics described in an accelerated coordinate frame

1. Correction from Lecture 3
2. Linear acceleration
3. Angular acceleration
4. Foucault pendulum



PHY 711 Classical Mechanics and Mathematical Methods

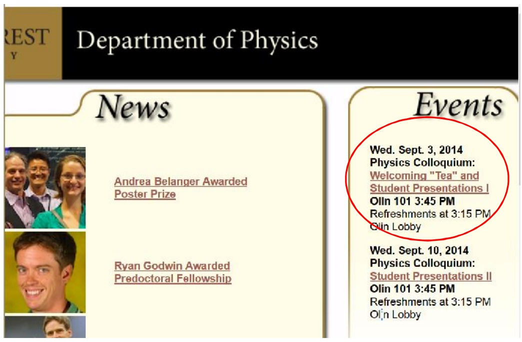
MWF 10 AM-10:50 PM	OPL 103	http://www.wfu.edu/~natalie/f14phy711/

Instructor: Natalie Holzwarth	Phone:758-5510	Office:300 OPL	e-mail:natalie@wfu.edu

Course schedule

(Preliminary schedule -- subject to frequent adjustment.) rrr>

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/27/2014	Chap. 1	Review of basic principles	#1
2	Fri, 8/29/2014	Chap. 1	Scattering theory	#2
3	Mon, 9/01/2014	Chap. 1	Scattering theory continued	#3
➡ 4	Wed, 9/03/2014	Chap. 2	Accelerated coordinate systems	#4



Department of Physics

News

Andrea Belanger Awarded Poster Prize

Ryan Godwin Awarded Predoctoral Fellowship

Events

Wed. Sept. 3, 2014
Physics Colloquium:
Welcoming "Tea" and Student Presentations I
Olin 101 3:45 PM
Refreshments at 3:15 PM
Olin Lobby

Wed. Sept. 10, 2014
Physics Colloquium:
Student Presentations II
Olin 101 3:45 PM
Refreshments at 3:15 PM
Olin Lobby

WFU Physics Colloquium

TITLE: "Welcome to the WFU Physics Department -- Part I"

TIME: Wednesday Sept. 3, 2014 at 3:45 PM*

PLACE: George P. Williams, Jr. Lecture Hall, (Olin 101)

* Note: early starting time.

Refreshments will be served at 3:15 PM in the lounge. All interested persons are cordially invited to attend.

PROGRAM

The purpose of this seminar is to help new, returning, and prospective students (including both undergraduate and graduate students), faculty, and staff to become acquainted with each other and with the Physics Department. After refreshments in the lounge in the lobby of Olin Physical Laboratory (starting at 3:15), we will meet in the George P. Williams, Jr. Lecture Hall (Olin 101) at 3:45 PM for some announcements followed by presentations by undergraduate students, highlighting their summer research experiences, and senior graduate students presenting overviews of their research areas. The student presentations will be continued next week.

Correction to last slide of Lecture 3:

$$\text{Rutherford scattering : } \quad V(r)=\frac{\kappa E}{r}=\frac{Z_1 Z_2 e^2}{r}$$

(Note that E is center of mass energy)

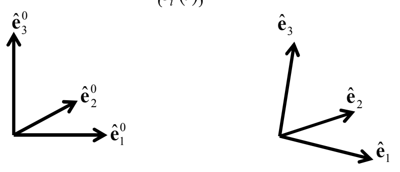
$$\left(\frac{d\sigma}{d\Omega}\right)\Bigg|_{CM}=\frac{\kappa^2}{16}\frac{1}{\sin^4(\theta/2)}$$

For $m_1 = m_2$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right)=\left(\frac{d\sigma_{CM}(2\psi)}{d\Omega_{CM}}\right)\cdot 4\cos\psi=\frac{\kappa^2}{4}\frac{\cos\psi}{\sin^4\psi}$$

Physical laws as described in non-inertial coordinate systems

- Newton's laws are formulated in an inertial frame of reference $\{\hat{\mathbf{e}}_i^0\}$
- For some problems, it is convenient to transform the the equations into a non-inertial coordinate system $\{\hat{\mathbf{e}}_i(t)\}$



Comparison of analysis in “inertial frame” versus “non-inertial frame”

Denote by $\hat{e}_i^0$ a fixed coordinate system

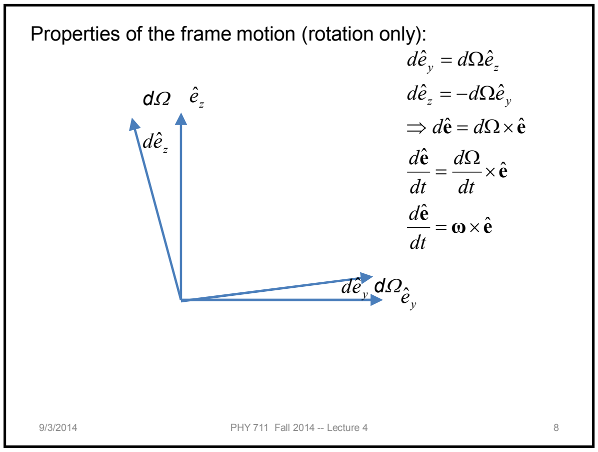
Denote by $\hat{e}_i$ a moving coordinate system

$$\mathbf{V} = \sum_{i=1}^{3} V_i^0 \hat{e}_i^0 = \sum_{i=1}^{3} V_i \hat{e}_i$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \sum_{i=1}^{3} \frac{dV_i^0}{dt}\hat{e}_i^0 = \sum_{i=1}^{3} \frac{dV_i}{dt}\hat{e}_i + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

$$\text{Define : } \left(\frac{d\mathbf{V}}{dt}\right)_{body} \equiv \sum_{i=1}^{3} \frac{dV_i}{dt}\hat{e}_i$$

$$\Rightarrow \left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$



Properties of the frame motion (rotation only):

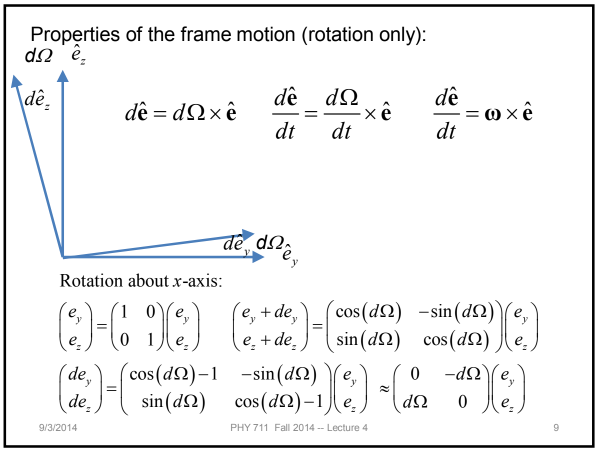
$$d\hat{e}_y = d\Omega \hat{e}_z$$

$$d\hat{e}_z = -d\Omega \hat{e}_y$$

$$\Rightarrow d\hat{\mathbf{e}} = d\mathbf{\Omega} \times \hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \frac{d\mathbf{\Omega}}{dt} \times \hat{\mathbf{e}}$$

$$\frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$



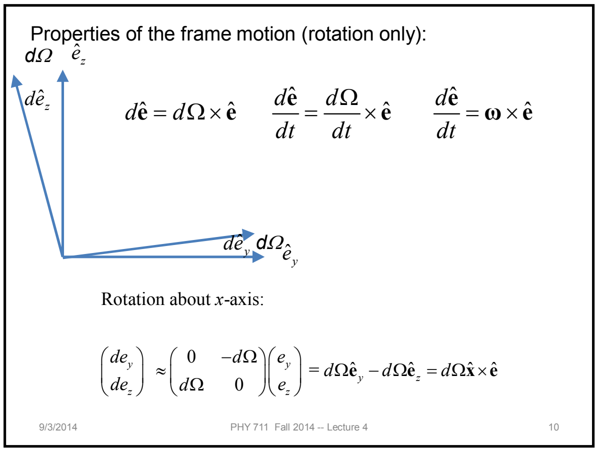
Properties of the frame motion (rotation only):

$$d\hat{\mathbf{e}} = d\mathbf{\Omega} \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \frac{d\mathbf{\Omega}}{dt} \times \hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

Rotation about x-axis:

$$\begin{pmatrix} e_y \\ e_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e_y \\ e_z \end{pmatrix} \qquad \begin{pmatrix} e_y + de_y \\ e_z + de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega) & -\sin(d\Omega) \\ \sin(d\Omega) & \cos(d\Omega) \end{pmatrix}\begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} = \begin{pmatrix} \cos(d\Omega)-1 & -\sin(d\Omega) \\ \sin(d\Omega) & \cos(d\Omega)-1 \end{pmatrix}\begin{pmatrix} e_y \\ e_z \end{pmatrix} \approx \begin{pmatrix} 0 & -d\Omega \\ d\Omega & 0 \end{pmatrix}\begin{pmatrix} e_y \\ e_z \end{pmatrix}$$

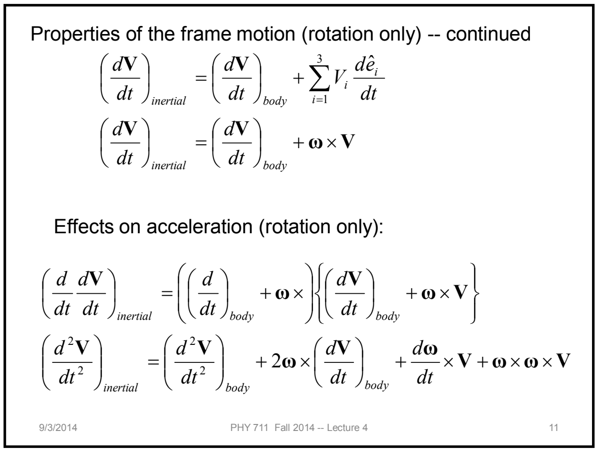


Properties of the frame motion (rotation only):

$$d\hat{\mathbf{e}} = d\boldsymbol{\Omega}\times\hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \frac{d\boldsymbol{\Omega}}{dt}\times\hat{\mathbf{e}} \qquad \frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega}\times\hat{\mathbf{e}}$$

Rotation about x-axis:

$$\begin{pmatrix} de_y \\ de_z \end{pmatrix} \approx \begin{pmatrix} 0 & -d\Omega \\ d\Omega & 0 \end{pmatrix}\begin{pmatrix} e_y \\ e_z \end{pmatrix} = d\Omega\hat{\mathbf{e}}_y - d\Omega\hat{\mathbf{e}}_z = d\Omega\hat{\mathbf{x}}\times\hat{\mathbf{e}}$$



Properties of the frame motion (rotation only) -- continued

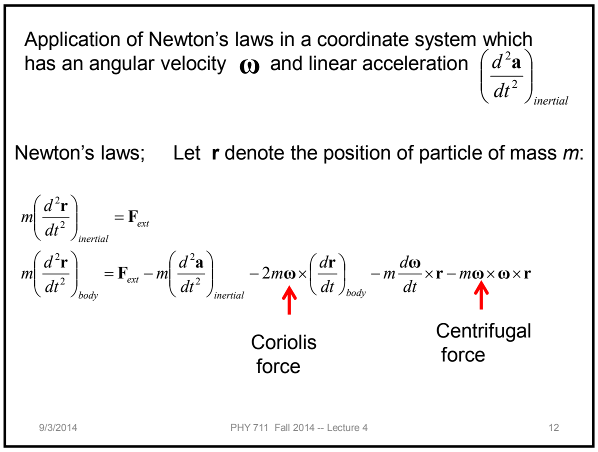
$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{V}$$

Effects on acceleration (rotation only):

$$\left(\frac{d}{dt}\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega}\times\right)\left\{\left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{V}\right\}$$

$$\left(\frac{d^2\mathbf{V}}{dt^2}\right)_{inertial} = \left(\frac{d^2\mathbf{V}}{dt^2}\right)_{body} + 2\boldsymbol{\omega}\times\left(\frac{d\mathbf{V}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{V} + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{V}$$



Application of Newton's laws in a coordinate system which has an angular velocity $\boldsymbol{\omega}$ and linear acceleration $\left(\frac{d^2\mathbf{a}}{dt^2}\right)_{inertial}$

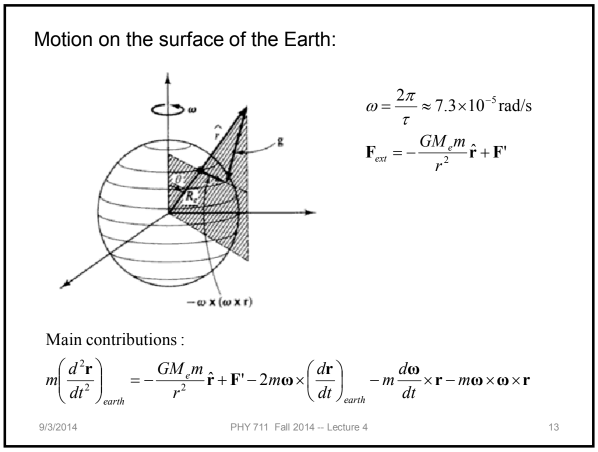
Newton's laws; Let **r** denote the position of particle of mass *m*:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{inertial} = \mathbf{F}_{ext}$$

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{body} = \mathbf{F}_{ext} - m\left(\frac{d^2\mathbf{a}}{dt^2}\right)_{inertial} - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{body} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

Coriolis force

Centrifugal force



Motion on the surface of the Earth:

$$\omega = \frac{2\pi}{\tau} \approx 7.3\times 10^{-5}\,\text{rad/s}$$

$$\mathbf{F}_{ext} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}'$$

Main contributions :

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

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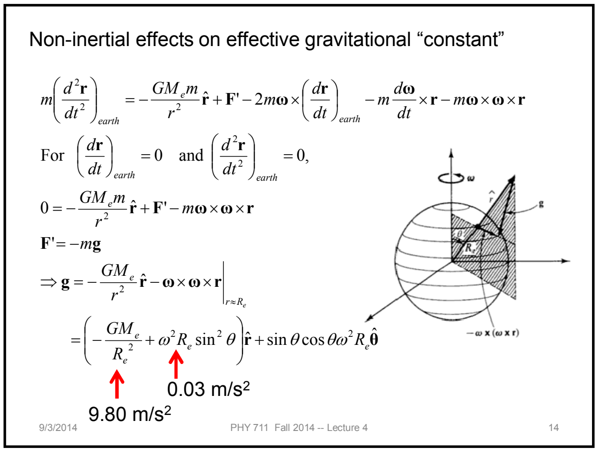

Non-inertial effects on effective gravitational "constant"

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - 2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

For $\left(\frac{d\mathbf{r}}{dt}\right)_{earth} = 0$ and $\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = 0,$

$$0 = -\frac{GM_e m}{r^2}\hat{\mathbf{r}} + \mathbf{F}' - m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$

$$\mathbf{F}' = -m\mathbf{g}$$

$$\Rightarrow \mathbf{g} = -\frac{GM_e}{r^2}\hat{\mathbf{r}} - \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}\Big|_{r\approx R_e}$$

$$= \left(-\underset{\uparrow\ 9.80\ \text{m/s}^2}{\frac{GM_e}{R_e{}^2}} + \underset{\uparrow\ 0.03\ \text{m/s}^2}{\omega^2 R_e \sin^2\theta}\right)\hat{\mathbf{r}} + \sin\theta\cos\theta\,\omega^2 R_e\hat{\boldsymbol{\theta}}$$

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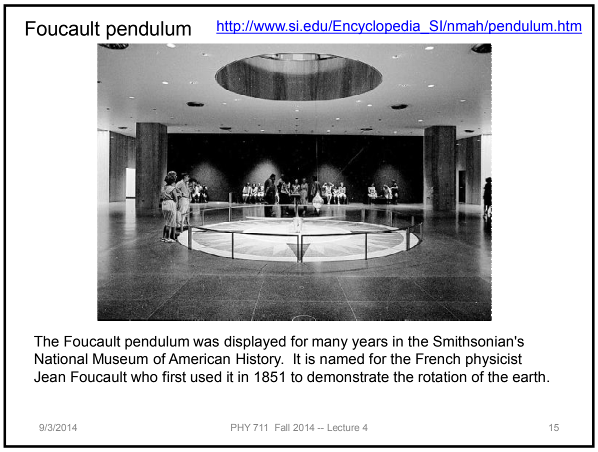

Foucault pendulum

http://www.si.edu/Encyclopedia_SI/nmah/pendulum.htm

The Foucault pendulum was displayed for many years in the Smithsonian's National Museum of American History. It is named for the French physicist Jean Foucault who first used it in 1851 to demonstrate the rotation of the earth.

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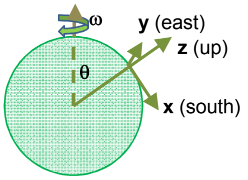
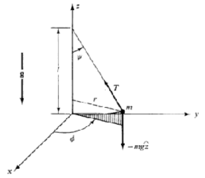
Equation of motion on Earth’s surface

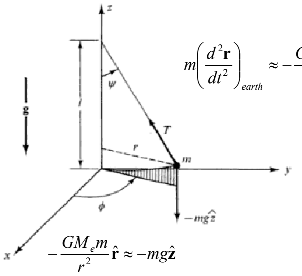
$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth}=-\frac{GM_e m}{r^2}\hat{\mathbf{r}}+\mathbf{F}'-2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}-m\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r}-m\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$$





$$\boldsymbol{\omega}\approx-\omega\sin\theta\hat{\mathbf{x}}+\omega\cos\theta\hat{\mathbf{z}}$$

Foucault pendulum continued – keeping leading terms:



$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth}\approx-\frac{GM_e m}{R_e^{\ 2}}\hat{\mathbf{r}}+\mathbf{F}'-2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}$$

$$-\frac{GM_e m}{r^2}\hat{\mathbf{r}}\approx-mg\hat{\mathbf{z}}$$

$$\mathbf{F}'\approx-T\sin\psi\cos\phi\hat{\mathbf{x}}-T\sin\psi\sin\phi\hat{\mathbf{y}}+T\cos\psi\hat{\mathbf{z}}$$

$$\boldsymbol{\omega}\approx-\omega\sin\theta\hat{\mathbf{x}}+\omega\cos\theta\hat{\mathbf{z}}$$

$$\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}\approx\omega\left(-\dot{y}\cos\theta\hat{\mathbf{x}}+\left(\dot{x}\cos\theta+\dot{z}\sin\theta\right)\hat{\mathbf{y}}-\dot{y}\sin\theta\hat{\mathbf{z}}\right)$$

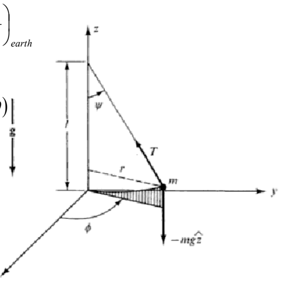
Foucault pendulum continued – keeping leading terms:

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth}\approx-\frac{GM_e m}{R_e^{\ 2}}\hat{\mathbf{r}}+\mathbf{F}'-2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}$$

$$m\ddot{x}\approx-T\sin\psi\cos\phi+2m\omega\dot{y}\cos\theta$$

$$m\ddot{y}\approx-T\sin\psi\sin\phi-2m\omega\left(\dot{x}\cos\theta+\dot{z}\sin\theta\right)$$

$$m\ddot{z}\approx T\cos\psi-mg+2m\omega\dot{y}\sin s\theta$$



Further approximation :

$\psi<<1;\quad \ddot{z}\approx 0;\quad T\approx mg$

$m\ddot{x}\approx-mg\sin\psi\cos\phi+2m\omega\dot{y}\cos\theta$

$m\ddot{y}\approx-mg\sin\psi\sin\phi-2m\omega\dot{x}\cos\theta$

Also note that :

$x\approx\ell\sin\psi\cos\phi$

$y\approx\ell\sin\psi\sin\phi$

Foucault pendulum continued – coupled equations:

$$\ddot{x} \approx -\frac{g}{\ell}x + 2\omega\cos\theta\dot{y}$$

$$\ddot{y} \approx -\frac{g}{\ell}y - 2\omega\cos\theta\dot{x}$$

Try to find a solution of the form:

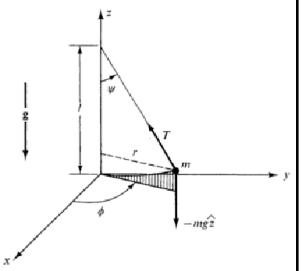
$x(t) = Xe^{-iqt} \quad y(t) = Ye^{-iqt}$

Denote $\omega_\perp \equiv \omega\cos\theta$

$$\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_\perp q \\ -i2\omega_\perp q & -q^2 + \frac{g}{\ell} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

Non - trivial solutions:

$$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp{}^2 + \frac{g}{\ell}}$$



Foucault pendulum continued – coupled equations:

Solution continued:

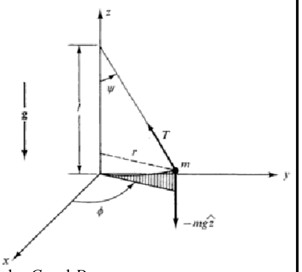
$x(t) = Xe^{-iqt} \quad y(t) = Ye^{-iqt}$

$$\begin{pmatrix} -q^2 + \frac{g}{\ell} & i2\omega_\perp q \\ -i2\omega_\perp q & -q^2 + \frac{g}{\ell} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

Non - trivial solutions:

$$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp{}^2 + \frac{g}{\ell}}$$

Amplitude relationship: $X = iY$



General solution with complex amplitudes C and D:

$$x(t) = \mathrm{Re}\left\{iCe^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t}\right\}$$

$$y(t) = \mathrm{Re}\left\{Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t}\right\}$$

General solution with complex amplitudes C and D:

$$x(t) = \mathrm{Re}\left\{iCe^{-i(\alpha+\beta)t} + iDe^{-i(\alpha-\beta)t}\right\}$$

$$y(t) = \mathrm{Re}\left\{Ce^{-i(\alpha+\beta)t} + De^{-i(\alpha-\beta)t}\right\}$$

$$q_\pm = \alpha \pm \beta \equiv \omega_\perp \pm \sqrt{\omega_\perp{}^2 + \frac{g}{\ell}} \approx \omega_\perp \pm \sqrt{\frac{g}{\ell}}$$

since $\omega_\perp \approx 7\times 10^{-5}\cos\theta \; rad/s << \sqrt{\frac{g}{\ell}}$

Suppose: $x(0) = X_0 \qquad y(0) = 0$

$$x(t) = X_0\cos\left(\sqrt{\tfrac{g}{\ell}}t\right)\cos\left(\omega_\perp t\right)$$

$$y(t) = -X_0\cos\left(\sqrt{\tfrac{g}{\ell}}t\right)\sin\left(\omega_\perp t\right)$$