PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 5:

Start reading Chapter 3 –

First focusing on the "calculus of variation"

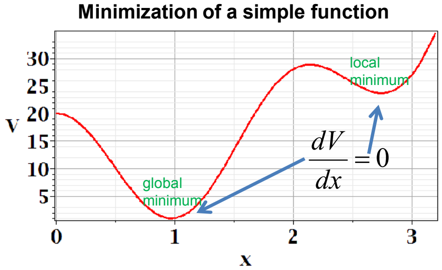
Course schedule

(Preliminary schedule -- subject to frequent adjustment.) rrr>

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/27/2014	Chap. 1	Review of basic principles	#1
2	Fri, 8/29/2014	Chap. 1	Scattering theory	#2
3	Mon, 9/01/2014	Chap. 1	Scattering theory continued	#3
4	Wed, 9/03/2014	Chap. 2	Accelerated coordinate systems	#4
5	Fri, 9/05/2014	Chap. 3	Calculus of variations	#5
6	Mon, 9/08/2014			

In Chapter 3, the notion of Lagrangian dynamics is developed; reformulating Newton's laws in terms of minimization of related functions. In preparation, we need to develop a mathematical tool known as "the calculus of variation".

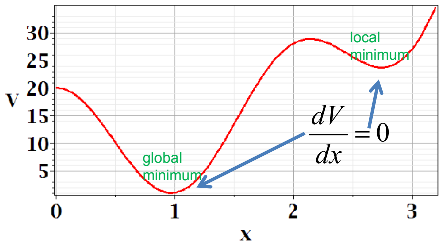
Minimization of a simple function



Minimization of a simple function

Given a function $V(x)$, find the value(s) of x for which $V(x)$ is minimized (or maximized).

Necessary condition : $\frac{dV}{dx}=0$



Functional minimization

Consider a family of functions $y(x)$, with the end points

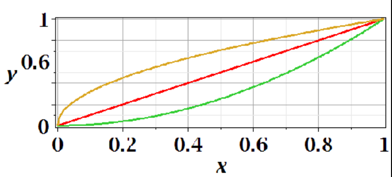
$y(x_i)=y_i$ and $y(x_f)=y_f$ and a function $L\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$.

Find the function $y(x)$ which extremizes $L\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$.

Necessary condition : $\delta L=0$

Example :

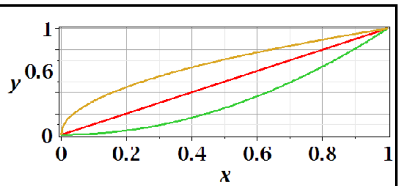
$$L=\int_{(0,0)}^{1,1}\sqrt{(dx)^2+(dy)^2}$$



Example :

$$L=\int_{(0,0)}^{1,1}\sqrt{(dx)^2+(dy)^2}$$

$$=\int_0^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$



Sample functions :

$y_1(x)=\sqrt{x}$ $\qquad L=\int_0^1\sqrt{1+\frac{1}{4x}}dx=1.4789$

$y_2(x)=x$ $\qquad L=\int_0^1\sqrt{1+1}dx=\sqrt{2}=1.4142$

$y_2(x)=x^2$ $\qquad L=\int_0^1\sqrt{1+4x^2}dx=1.4789$

Calculus of variation example for a pure integral functions

Find the function $y(x)$ which extremizes $L\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$

where $L\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) \equiv \int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx.$

Necessary condition : $\delta L = 0$

At any x, let $y(x) \rightarrow y(x) + \delta y(x)$

$$\frac{dy(x)}{dx} \rightarrow \frac{dy(x)}{dx} + \delta\frac{dy(x)}{dx}$$

Formally :

$$\delta L = \int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}\delta y + \left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta\left(\frac{dy}{dx}\right)\right]\right]dx.$$

9/5/2014 PHY 711 Fall 2014 -- Lecture 5 7

After some derivations, we find

$$\delta L = \int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}}\delta y + \left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\delta\left(\frac{dy}{dx}\right)\right]\right]dx$$

$$= \int_{x_i}^{x_f}\left[\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right]\right]\delta y\, dx = 0 \quad \text{for all } x_i \le x \le x_f$$

$$\Rightarrow \left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0 \quad \text{for all } x_i \le x \le x_f$$

9/5/2014 PHY 711 Fall 2014 -- Lecture 5 8

Example : End points -- $y(0) = 0;\ y(1) = 1$

$$L = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \Rightarrow f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right) = 0$$

Solution :

$$\left(\frac{dy/dx}{\sqrt{1+(dy/dx)^2}}\right) = K \qquad \frac{dy}{dx} = K' \equiv \frac{K}{\sqrt{1-K^2}}$$

$$\Rightarrow y(x) = x$$

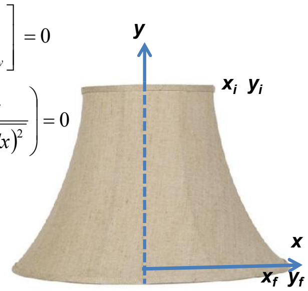
9/5/2014 PHY 711 Fall 2014 -- Lecture 5 9

Example :

$$A = 2\pi \int_{x_i}^{x_f} x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \Rightarrow f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x, \frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)_{x,y}\right] = 0$$

$$\Rightarrow -\frac{d}{dx}\left(\frac{x dy/dx}{\sqrt{1 + (dy/dx)^2}}\right) = 0$$



$$-\frac{d}{dx}\left(\frac{x dy/dx}{\sqrt{1 + (dy/dx)^2}}\right) = 0$$

$$\frac{x dy/dx}{\sqrt{1 + (dy/dx)^2}} = K_1$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{\left(\frac{x}{K_1}\right)^2 - 1}}$$

$$\Rightarrow y(x) = K_2 - K_1 \ln\left(x + \sqrt{x^2 - K_1^{\,2}}\right)$$

Another example:
(Courtesy of F. B. Hildebrand, Methods of Applied Mathematics)

Consider all curves $y(x)$ with $y(0) = 0$ and $y(1) = 1$
that minimize the integral :

$$I = \int_0^1 \left(\left(\frac{dy}{dx}\right)^2 - ay^2\right) dx \quad \text{for constant } a > 0$$

Euler - Lagrange equation :

$$\frac{d^2 y}{dx^2} + ay = 0$$

$$\Rightarrow y(x) = \frac{\sin\left(\sqrt{a}x\right)}{\sin\left(\sqrt{a}\right)}$$

Review : for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

a necessary condition to extremize $\int_{x_i}^{x_f} f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)dx$:

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0$$

Note that for $f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right)$,

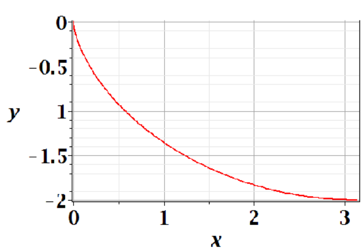
$$\frac{df}{dx} = \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial(dy/dx)}\right)\right)\frac{dy}{dx} + \left(\frac{\partial f}{\partial(dy/dx)}\right)\frac{d}{dx}\frac{dy}{dx} + \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Brachistochrone problem: (solved by Newton in 1696)

http://mathworld.wolfram.com/BrachistochroneProblem.html



A particle of weight *mg* travels frictionlessly down a path of shape *y(x)*. What is the shape of the path *y(x)* that minimizes the travel time from *y(0)=0* to *y(π)=-2*?

$$T = \int_{x_i y_i}^{x_f y_f}\frac{ds}{v} = \int_{x_i}^{x_f}\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{2gy}}dx \quad \text{because} \quad \tfrac{1}{2}mv^2 = mgy$$

$$f\left(\left\{y(x),\frac{dy}{dx}\right\},x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = 0$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

Note that for the original form of Euler - Lagrange equation :

$$\left(\frac{\partial f}{\partial y}\right)_{x,\frac{dy}{dx}} - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)_{x,y}\right] = 0,$$

differential equation is more complicated :

$$-\frac{1}{2}\sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{y^3}} - \frac{d}{dx}\left(\frac{\frac{dy}{dx}}{\sqrt{y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0$$

$$T = \int_{x_i y_i}^{x_f y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{-2gy}}\,dx \quad \text{because} \quad \tfrac{1}{2}mv^2 = -mgy$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

Alternative relationships for extremization:

$$\left(\frac{\partial f}{\partial y}\right) - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial(dy/dx)}\right)\right] = 0$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial(dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{-y\left(1+\left(\frac{dy}{dx}\right)^2\right)}}\right) = 0 \qquad -y\left(1+\left(\frac{dy}{dx}\right)^2\right) = K \equiv 2a$$

$$-y\left(1+\left(\frac{dy}{dx}\right)^2\right) = K \equiv 2a \qquad \text{Let} \quad y = -2a\sin^2\tfrac{\theta}{2} = a(\cos\theta - 1)$$

$$\frac{dy}{dx} = -\sqrt{\frac{2a}{-y} - 1}$$

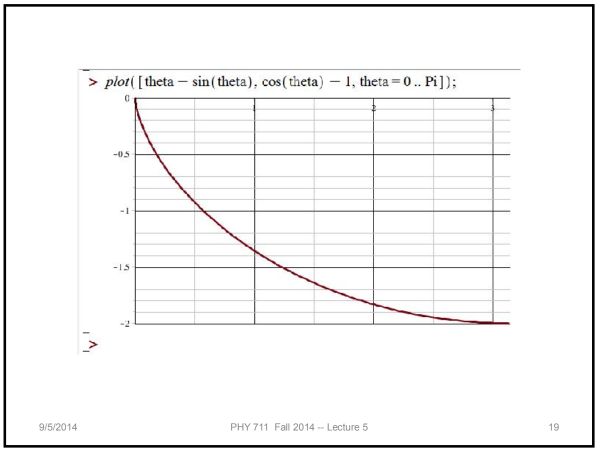
$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}} = dx$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}-1}} = \frac{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta}{\sqrt{\frac{2a}{2a\sin^2\frac{\theta}{2}}-1}} = dx$$

$$x = \int_0^\theta a(1-\cos\theta')d\theta' = a(\theta - \sin\theta)$$

Parametric equations for Brachistochrone:

$$x = a(\theta - \sin\theta)$$
$$y = a(\cos\theta - 1)$$


> plot([theta − sin(theta), cos(theta) − 1, theta = 0 .. Pi]);
0
1
2
3
-0.5
-1
-1.5
-2
9/5/2014
PHY 711 Fall 2014 -- Lecture 5
19