

**PHY 711 Classical Mechanics and
Mathematical Methods**
10-10:50 AM MWF Olin 103

Plan for Lecture 6:

Continue reading Chapter 3

**Further development of the
“calculus of variation”**

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PHY 711 Classical Mechanics and Mathematical Method

MWF 10 AM-10:50 PM OPL 103 <http://www.wfu.edu/~natalie/f14phy711/>

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.) rrr>

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/27/2014	Chap. 1	Review of basic principles	#1
2	Fri, 8/29/2014	Chap. 1	Scattering theory	#2
3	Mon, 9/01/2014	Chap. 1	Scattering theory continued	#3
4	Wed, 9/03/2014	Chap. 2	Accelerated coordinate systems	#4
5	Fri, 9/05/2014	Chap. 3	Calculus of variations	#5
6	Mon, 9/08/2014	Chap. 3	Calculus of variations	#6

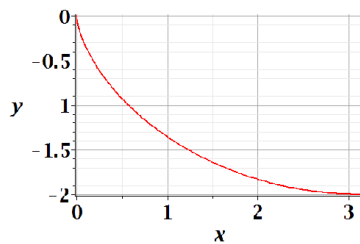
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Brachistochrone problem: (solved by Newton in 1696)

<http://mathworld.wolfram.com/BrachistochroneProblem.html>



A particle of weight mg travels frictionlessly down a path of shape $y(x)$. What is the shape of the path $y(x)$ that minimizes the travel time from $y(0)=0$ to $y(\pi)=-2$?

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$$T = \int_{x_i, y_i}^{x_f, y_f} \frac{ds}{v} = \int_{x_i}^{x_f} \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-2gy}} dx \quad \text{because } \frac{1}{2}mv^2 = -mgy$$

$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-y}}$$

Alternative relationships for extremization:

$$\left(\frac{\partial f}{\partial y}\right) - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right) \right] = 0$$

$$\frac{d}{dx} \left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x} \right)$$

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$$f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) = \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-y}}$$

$$\frac{d}{dx} \left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x} \right)$$

$$\Rightarrow \frac{d}{dx} \left(\frac{1}{\sqrt{-y \left(1 + \left(\frac{dy}{dx} \right)^2 \right)}} \right) = 0 \quad -y \left(1 + \left(\frac{dy}{dx} \right)^2 \right) = K \equiv 2a$$

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$$-y \left(1 + \left(\frac{dy}{dx} \right)^2 \right) = K \equiv 2a \quad \text{Let } y = -2a \sin^2 \frac{\theta}{2} = a(\cos \theta - 1)$$

$$\frac{dy}{dx} = -\frac{\sqrt{2a}}{\sqrt{-y}} \quad -\frac{dy}{\sqrt{\frac{2a}{-y}} - 1} = \frac{2a \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta}{\sqrt{\frac{2a}{2a \sin^2 \frac{\theta}{2}} - 1}} = dx$$

$$-\frac{dy}{\sqrt{\frac{2a}{-y}} - 1} = dx \quad x = \int_0^\theta a(1 - \cos \theta') d\theta' = a(\theta - \sin \theta)$$

Parametric equations for Brachistochrone:

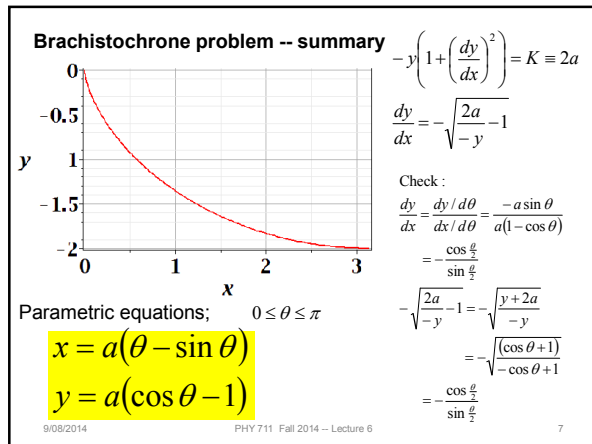
$$x = a(\theta - \sin \theta)$$

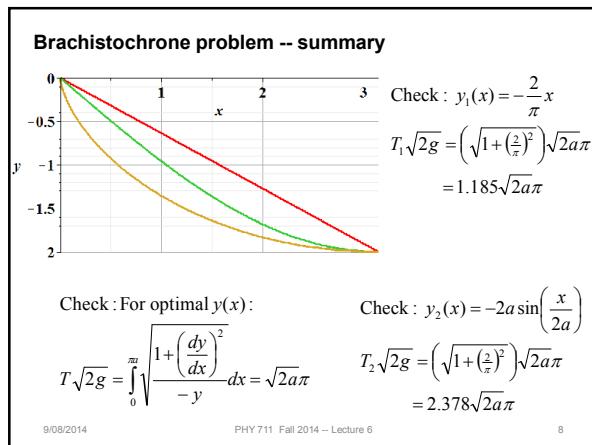
$$y = a(\cos \theta - 1)$$

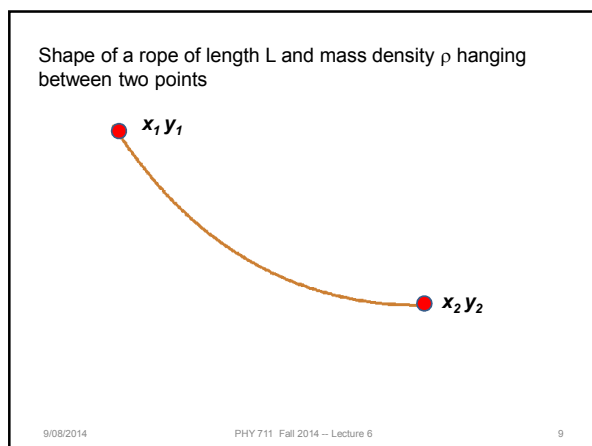
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Potential energy of hanging rope :

$$E = \rho g \int_{x_1}^{x_2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Length of rope :

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Define a composite function to minimize :

$$W \equiv E + \lambda L$$

Lagrange multiplier

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$$W = \int_{x_1}^{x_2} (\rho g y + \lambda) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$f\left(y, \frac{dy}{dx}\right) = (\rho g y + \lambda) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{d}{dx} \left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x} \right)$$

$$\Rightarrow (\rho g y + \lambda) \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right) = K$$

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$$(\rho g y + \lambda) \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - \frac{\left(\frac{dy}{dx}\right)^2}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \right) = K$$

$$(\rho g y + \lambda) \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = K$$

$$y(x) = -\frac{1}{\rho g} \left(\lambda + K \cosh\left(\frac{x-a}{K/\rho g}\right) \right)$$

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$$y(x) = -\frac{1}{\rho g} \left(\lambda + K \cosh \left(\frac{x-a}{K/\rho g} \right) \right)$$

Integration constants : K, a, λ

Constraints : $y(x_1) = y_1$

$y(x_2) = y_2$

$$\int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx = L$$

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Summary of results

For the class of problems where we need to perform an extremization on an integral form :

$$I = \int_{x_i}^{x_f} f \left(y(x), \frac{dy}{dx}, x \right) dx \quad \delta I = 0$$

A necessary condition is the Euler - Lagrange equations :

$$\left(\frac{\partial f}{\partial y} \right) - \frac{d}{dx} \left[\left(\frac{\partial f}{\partial (dy/dx)} \right) \right] = 0$$

$$\frac{d}{dx} \left(f - \frac{\partial f}{\partial (dy/dx)} \frac{dy}{dx} \right) = \left(\frac{\partial f}{\partial x} \right)$$

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Application to particle dynamics

$x \rightarrow t$ (time)

$y \rightarrow q$ (generalized coordinate)

$f \rightarrow L$ (Lagrangian)

$I \rightarrow A$ (action)

Denote : $\dot{q} \equiv \frac{dq}{dt}$

$$A = \int_{t_1}^{t_2} L(q, \dot{q}; t) dt$$

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