PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 8:

Continue reading Chapter 3 & 6

1. **D'Alembert's principle**
2. **Hamilton's principle**
3. **Lagrange's equations in presence of magnetic fields**

PHY 711 Classical Mechanics and Mathematical Methods

MWF 10 AM-10:50 PM OPL 103 http://www.wfu.edu/~natalie/f14phy711/

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.) rrr>

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/27/2014	Chap. 1	Review of basic principles	#1
2	Fri, 8/29/2014	Chap. 1	Scattering theory	#2
3	Mon, 9/01/2014	Chap. 1	Scattering theory continued	#3
4	Wed, 9/03/2014	Chap. 2	Accelerated coordinate systems	#4
5	Fri, 9/05/2014	Chap. 3	Calculus of variations	#5
6	Mon, 9/08/2014	Chap. 3	Calculus of variations	#6
7	Wed, 9/10/2014	Chap. 3	Hamilton's principle	#7
8	Fri, 9/12/2014	Chap. 3 & 6	Hamilton's principle	#8
9	Mon, 9/15/2014			

Hamilton's principle:

Given the Lagrangian function: $L = L(\{q_\sigma\},\{\dot{q}_\sigma\},t) \equiv T - U,$

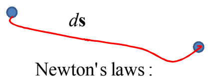
The physical trajectories of the generalized coordinates $\{q_\sigma(t)\}$

Are those which minimize the action: $S = \int L(\{q_\sigma\},\{\dot{q}_\sigma\},t)\,dt$

Euler-Lagrange equations:

$$\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right)\delta q_\sigma = 0 \quad \Rightarrow \text{for each } \sigma: \quad \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0$$

D'Alembert's principle -- notion of virtual work:



Generalized coordinates: $q_\sigma(\{x_i\})$

Newton's laws:

$$\mathbf{F}\text{-}m\mathbf{a}=0 \qquad \Rightarrow (\mathbf{F}\text{-}m\mathbf{a})\cdot d\mathbf{s}=0$$

$$m\mathbf{a}\cdot d\mathbf{s}=\sum_\sigma\sum_i m\ddot{x}_i\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma$$

$$=\sum_\sigma\sum_i\left(\frac{d}{dt}\left(m\dot{x}_i\frac{\partial x_i}{\partial q_\sigma}\right)-m\dot{x}_i\frac{d}{dt}\frac{\partial x_i}{\partial q_\sigma}\right)\delta q_\sigma$$

Claim: $\frac{\partial x_i}{\partial q_\sigma}=\frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$ and $\frac{d}{dt}\frac{\partial x_i}{\partial q_\sigma}=\frac{\partial}{\partial q_\sigma}\frac{dx_i}{dt}\equiv\frac{\partial \dot{x}_i}{\partial q_\sigma}$

$$m\mathbf{a}\cdot d\mathbf{s}=\sum_\sigma\sum_i\left(\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}m\dot{x}_i^2\right)}{\partial\dot{q}_\sigma}\right)-\frac{\partial\left(\frac{1}{2}m\dot{x}_i^2\right)}{\partial q_\sigma}\right)\delta q_\sigma$$

Some details --

$$m\mathbf{a}\cdot d\mathbf{s}=\sum_\sigma\sum_i m\ddot{x}_i\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma \qquad \text{where} \quad x_i(\{q_\sigma(t)\},t)$$

$$=\sum_\sigma\sum_i\left(\frac{d}{dt}\left(m\dot{x}_i\frac{\partial x_i}{\partial q_\sigma}\right)-m\dot{x}_i\frac{d}{dt}\frac{\partial x_i}{\partial q_\sigma}\right)\delta q_\sigma$$

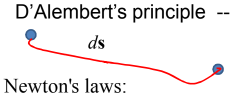
Claim: $\frac{\partial x_i}{\partial q_\sigma}=\frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$ and $\frac{d}{dt}\frac{\partial x_i}{\partial q_\sigma}=\frac{\partial}{\partial q_\sigma}\frac{dx_i}{dt}\equiv\frac{\partial \dot{x}_i}{\partial q_\sigma}$

Details: $\frac{dx_i}{dt}=\sum_\sigma\frac{\partial x_i}{\partial q_\sigma}\frac{dq_\sigma}{dt}+\frac{\partial x_i}{\partial t}=\sum_\sigma\frac{\partial x_i}{\partial q_\sigma}\dot{q}_\sigma+\frac{\partial x_i}{\partial t}$

$$\Rightarrow \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}=\frac{\partial x_i}{\partial q_\sigma}$$

$$\Rightarrow \frac{\partial \dot{x}_i}{\partial q_\sigma}=\sum_{\sigma'}\frac{\partial^2 x_i}{\partial q_\sigma\partial q_{\sigma'}}\dot{q}_{\sigma'}+\frac{\partial^2 x_i}{\partial q_\sigma\partial t}=\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_\sigma}\right)$$

D'Alembert's principle -- notion of virtual work:



Generalized coordinates: $q_\sigma(\{x_i\})$

Newton's laws:

$$\mathbf{F}\text{ - }m\mathbf{a}=0 \qquad \Rightarrow (\mathbf{F}\text{ - }m\mathbf{a})\cdot d\mathbf{s}=0$$

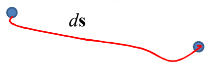
$$m\mathbf{a}\cdot d\mathbf{s}=\sum_\sigma\sum_i\left(\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}m\dot{x}_i^2\right)}{\partial\dot{q}_\sigma}\right)-\frac{\partial\left(\frac{1}{2}m\dot{x}_i^2\right)}{\partial q_\sigma}\right)\delta q_\sigma$$

$$=\sum_\sigma\left(\frac{d}{dt}\left(\frac{\partial(T)}{\partial\dot{q}_\sigma}\right)-\frac{\partial(T)}{\partial q_\sigma}\right)\delta q_\sigma \qquad \text{where } T\equiv\sum_i\left(\frac{1}{2}m\dot{x}_i^2\right)$$

$$\mathbf{F}\cdot d\mathbf{s}=\sum_\sigma\sum_i F_i\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma=-\sum_\sigma\sum_i\frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma=-\sum_\sigma\frac{\partial U}{\partial q_\sigma}\delta q_\sigma$$

$$(\mathbf{F}\text{ - }m\mathbf{a})\cdot d\mathbf{s}=-\sum_\sigma\left(\frac{d}{dt}\left(\frac{\partial(T-U)}{\partial\dot{q}_\sigma}\right)-\frac{\partial(T-U)}{\partial q_\sigma}\right)\delta q_\sigma=0$$

D'Alembert's principle -- notion of virtual work:



Generalized coordinates : $q_\sigma(\{x_i\})$

$$(\mathbf{F} - m\mathbf{a}) \cdot d\mathbf{s} = -\sum_\sigma \left(\frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_\sigma} \right) - \frac{\partial (T-U)}{\partial q_\sigma} \right) \delta q_\sigma = 0$$

Provided that $\frac{\partial U}{\partial \dot{q}_\sigma} = 0$

Consistent with Hamilton's principle with

$$L = T - U = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)$$

Example – simple harmonic oscillator

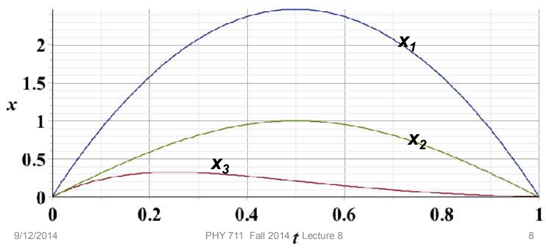
$T = \frac{1}{2} m \dot{x}^2 \qquad U = \frac{1}{2} m\omega^2 x^2$

Assume $x(0) = 0$ and $x(\frac{\pi}{\omega}) = 0 \qquad S = \int_0^{\pi/\omega} (T-U)\,dt$

Trial functions $x_1(t) = A\sin(\omega t) \qquad S_1 = 0$

$x_2(t) = A\omega t \cdot (\pi - \omega t) \qquad S_2 = 0.067 A^2 m\omega^2$

$x_3(t) = Ae^{-\omega t}\sin(\omega t) \qquad S_3 = 0.062 A^2 m\omega^2$



Note: in "proof" of Hamilton's principle:

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} \right) = 0 \quad \text{for} \quad L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) \equiv T - U$$

It was necessary to assume that :

$\frac{d}{dt}\frac{\partial U}{\partial \dot{q}_\sigma}$ does not contribute to the result.

$\Rightarrow$ How can we represent velocity-dependent forces?

Lorentz forces:

For particle of charge q in an electric field $\mathbf{E}(\mathbf{r},t)$ and magnetic field $\mathbf{B}(\mathbf{r},t)$:

Lorentz force: $\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)$

x – component: $F_x = q\left(E_x + \frac{1}{c}(\mathbf{v}\times\mathbf{B})_x\right)$

In this case, it is convenient to use cartesian coordinates

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

x - component: $\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}\right) = 0$

Apparently: $F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$

Answer: $U = q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$

where $\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}$ $\quad \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t)$

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Lorentz forces, continued:

x – component of Lorentz force: $F_x = q\left(E_x + \frac{1}{c}(\mathbf{v}\times\mathbf{B})_x\right)$

Suppose: $U = q\Phi(\mathbf{r},t) - \frac{q}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t)$

Consider: $F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\left(\dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}A_x(\mathbf{r},t)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}\frac{dA_x(\mathbf{r},t)}{dt} = -\frac{q}{c}\left(\frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

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Lorentz forces, continued:

$$-\frac{\partial U}{\partial x} = -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\left(\dot{x}\frac{\partial A_x(\mathbf{r},t)}{\partial x} + \dot{y}\frac{\partial A_y(\mathbf{r},t)}{\partial x} + \dot{z}\frac{\partial A_z(\mathbf{r},t)}{\partial x}\right)$$

$$\frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c}\left(\frac{\partial A_x(\mathbf{r},t)}{\partial x}\dot{x} + \frac{\partial A_x(\mathbf{r},t)}{\partial y}\dot{y} + \frac{\partial A_x(\mathbf{r},t)}{\partial z}\dot{z} + \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t}$$

$$= -q\frac{\partial \Phi(\mathbf{r},t)}{\partial x} - \frac{q}{c}\frac{\partial A_x(\mathbf{r},t)}{\partial t} + \frac{q}{c}\dot{y}\left(\frac{\partial A_y(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial y}\right) + \frac{q}{c}\dot{z}\left(\frac{\partial A_z(\mathbf{r},t)}{\partial x} - \frac{\partial A_x(\mathbf{r},t)}{\partial z}\right)$$

$$= qE_x(\mathbf{r},t) + \frac{q}{c}\left(\dot{y}B_z(\mathbf{r},t) - \dot{z}B_y(\mathbf{r},t)\right) = qE_x(\mathbf{r},t) + \frac{q}{c}\left(\mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right)_x$$

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Lorentz forces, continued:

Summary of results (using cartesian coordinates)

$$L = L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \equiv T - U$$

$$T = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad U = q\Phi(\mathbf{r}, t) - \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

where $\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c}\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}$ $\qquad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t)$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi(\mathbf{r}, t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

Example Lorentz force

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - q\Phi(\mathbf{r}, t) + \frac{q}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

Suppose $\mathbf{E}(\mathbf{r}, t) \equiv 0, \quad \mathbf{B}(\mathbf{r}, t) \equiv B_0\hat{\mathbf{z}}$

$$\mathbf{A}(\mathbf{r}, t) = \tfrac{1}{2} B_0\left(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}\right)$$

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{x} - \frac{q}{2c}B_0 y\right) - \frac{q}{2c}B_0\dot{y} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \qquad \Rightarrow \frac{d}{dt}\left(m\dot{y} + \frac{q}{2c}B_0 x\right) + \frac{q}{2c}B_0\dot{x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0 \qquad \Rightarrow \frac{d}{dt}m\dot{z} = 0$$

Example Lorentz force -- continued

$$L = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{2c} B_0\left(-\dot{x}y + \dot{y}x\right)$$

$$\frac{d}{dt}\left(m\dot{x} - \frac{q}{2c}B_0 y\right) - \frac{q}{2c}B_0\dot{y} = 0 \qquad \Rightarrow m\ddot{x} - \frac{q}{c}B_0\dot{y} = 0$$

$$\frac{d}{dt}\left(m\dot{y} + \frac{q}{2c}B_0 x\right) + \frac{q}{2c}B_0\dot{x} = 0 \qquad \Rightarrow m\ddot{y} + \frac{q}{c}B_0\dot{x} = 0$$

$$\frac{d}{dt}m\dot{z} = 0 \qquad \Rightarrow m\ddot{z} = 0$$

Example Lorentz force -- continued

$$L=\tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{q}{2c}B_0\left(-\dot{x}y+\dot{y}x\right)$$

$$m\ddot{x}=+\frac{q}{c}B_0\dot{y}$$

$$m\ddot{y}=-\frac{q}{c}B_0\dot{x}$$

$$m\ddot{z}=0$$

Note that same equations are obtained from direct application of Newton's laws:

$$m\ddot{\mathbf{r}}=\frac{q}{c}\dot{\mathbf{r}}\times B_0\hat{\mathbf{z}}$$

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Example Lorentz force -- continued

Consider formulation with different Gauge: $\mathbf{A}(\mathbf{r})=-B_0 y\hat{\mathbf{x}}$

$$L=\tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)-\frac{q}{c}B_0\dot{x}y$$

$$\frac{d}{dt}\left(m\dot{x}-\frac{q}{c}B_0 y\right)=0 \qquad \Rightarrow m\ddot{x}-\frac{q}{c}B_0\dot{y}=0$$

$$\frac{d}{dt}(m\dot{y})+\frac{q}{c}B_0\dot{x}=0 \qquad \Rightarrow m\ddot{y}+\frac{q}{c}B_0\dot{x}=0$$

$$\frac{d}{dt}m\dot{z}=0 \qquad \Rightarrow m\ddot{z}=0$$

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Example Lorentz force -- continued

Evaluation of equations:

$$m\ddot{x}-\frac{q}{c}B_0\dot{y}=0$$

$$m\ddot{y}+\frac{q}{c}B_0\dot{x}=0$$

$$m\ddot{z}=0$$

$$\dot{x}(t)=V_0\sin\left(\tfrac{q}{mc}t+\varphi\right)$$

$$\dot{y}(t)=V_0\cos\left(\tfrac{q}{mc}t+\varphi\right)$$

$$\dot{z}(t)=V_{0z}$$

$$x(t)=x_0-\tfrac{mc}{q}V_0\cos\left(\tfrac{q}{mc}t+\varphi\right)$$

$$y(t)=y_0+\tfrac{mc}{q}V_0\sin\left(\tfrac{q}{mc}t+\varphi\right)$$

$$z(t)=z_0+V_{0z}t$$

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