PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 9:

Continue reading Chapter 3 & 6

1. **Summary & review**
2. **Lagrange's equations with constraints**

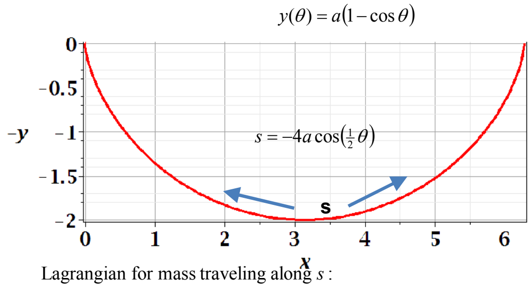
Course schedule

(Preliminary schedule -- subject to frequent adjustment.) rrr>

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/27/2014	Chap. 1	Review of basic principles	#1
2	Fri, 8/29/2014	Chap. 1	Scattering theory	#2
3	Mon, 9/01/2014	Chap. 1	Scattering theory continued	#3
4	Wed, 9/03/2014	Chap. 2	Accelerated coordinate systems	#4
5	Fri, 9/05/2014	Chap. 3	Calculus of variations	#5
6	Mon, 9/08/2014	Chap. 3	Calculus of variations	#6
7	Wed, 9/10/2014	Chap. 3	Hamilton's principle	#7
8	Fri, 9/12/2014	Chap. 3 & 6	Hamilton's principle	#8
9	Mon, 9/15/2014	Chap. 3 & 6	Lagrangians with constraints	#9
10	Wed, 9/17/2014			

Comment on problem Lagrangian formulation of Brachistochrone motion:

$$x(\theta) = a(\theta - \sin\theta)$$
$$y(\theta) = a(1 - \cos\theta)$$



Lagrangian for mass traveling along s :

$$L(s(t),\dot{s}(t)) = \tfrac{1}{2}m\dot{s}^2 - mgy = \tfrac{1}{2}m\dot{s}^2 - mg2a\left(1-\left(\frac{s}{4a}\right)^2\right)$$

Lagrangian for mass traveling along s :

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$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = 0$$

$$\Rightarrow m\ddot{s} = -\frac{mg}{4a}s$$

$$\Rightarrow \ddot{s} = -\frac{g}{4a}s$$

$$s(t) = s_0 + A\sin\left(\sqrt{\frac{g}{4a}}t\right)$$

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Comments on generalized coordinates:

$$L = L(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Here we have assumed that the generalized coordinates q_σ are independent. Now consider the possibility that the coordinates are related through constraint equations of the form:

Lagrangian : $L = L(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t)$

Constraints : $f_j = f_j(\{q_\sigma(t)\},t) = 0$

Lagrange multipliers

Modified Euler - Lagrange equations : $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$

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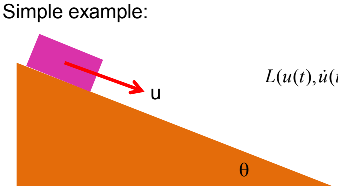
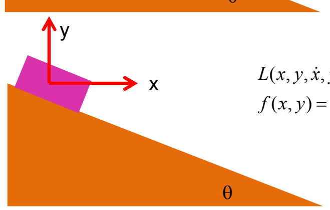
Simple example:



$$L(u(t),\dot{u}(t)) = \tfrac{1}{2}m\dot{u}^2 + mgu\sin\theta$$



$$L(x,y,\dot{x},\dot{y}) = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy$$

$$f(x,y) = \sin\theta\, x + \cos\theta\, y = 0$$

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Case 1:

$$L(u(t),\dot{u}(t)) = \tfrac{1}{2}m\dot{u}^2 + mgu\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = 0 = m\ddot{u} - mg\sin\theta = 0$$

$$\Rightarrow \ddot{u} = g\sin\theta$$

Case 2:

$$L(x,y,\dot{x},\dot{y}) = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$f(x,y) = \sin\theta\, x + \cos\theta\, y = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \lambda\frac{\partial f}{\partial x} = 0 = m\ddot{x} + \lambda\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \lambda\frac{\partial f}{\partial y} = 0 = m\ddot{y} + mg + \lambda\cos\theta$$

$$\sin\theta\,\ddot{x} + \cos\theta\,\ddot{y} = 0$$

$$\Rightarrow \lambda = mg\cos\theta$$

$$\left(-\cos\theta\,\ddot{x} + \sin\theta\,\ddot{y}\right) = -g\sin\theta$$

Rational for Lagrange multipliers

Recall Hamilton's principle:

$$S = \int_{t_i}^{t_f} L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)dt$$

$$\delta S = 0 = \int_{t_i}^{t_f}\left(\sum_\sigma\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right)\delta q_\sigma\right)dt$$

With constraints: $f_j = f_j\left(\{q_\sigma(t)\},t\right) = 0$

Variations δq_σ are no longer independent.

$$\delta f_j = 0 = \sum_\sigma \frac{\partial f_j}{\partial q_\sigma}\delta q_\sigma \quad \text{at each } t$$

$\Rightarrow$ Add 0 to Euler-Lagrange equations in the form:

$$\sum_j \lambda_j \sum_\sigma \frac{\partial f_j}{\partial q_\sigma}\delta q_\sigma$$

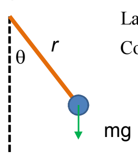
Euler-Lagrange equations with constraints:

Lagrangian: $L = L\left(\{q_\sigma(t)\},\{\dot{q}_\sigma(t)\},t\right)$

Constraints: $f_j = f_j\left(\{q_\sigma(t)\},t\right) = 0$

Modified Euler-Lagrange equations: $\displaystyle \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_\sigma} = 0$

Example:



Lagrangian: $L = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$

Constraints: $f = r - \ell = 0$

Example continued:

$$\text{Lagrangian}: \quad L = \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta$$

$$\text{Constraints}: \quad f = r - \ell = 0$$

$$\frac{d}{dt} m\dot{r} - mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$

$$\frac{d}{dt} mr^2\dot{\theta} + mgr\sin\theta = 0$$

$$\dot{r} = 0 = \ddot{r} \qquad r = \ell$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

$$\Rightarrow \lambda = m\ell\dot{\theta}^2 + mg\cos\theta$$

Another example:

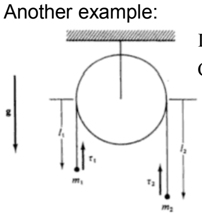


Figure 19.1 Atwood's machine.

$$\text{Lagrangian}: \quad L = \tfrac{1}{2} m_1\dot{\ell}_1^2 + \tfrac{1}{2} m_2\dot{\ell}_2^2 + m_1 g\ell_1 + m_2 g\ell_2$$

$$\text{Constraints}: \quad f = \ell_1 + \ell_2 - \ell = 0$$

$$\frac{d}{dt} m_1\dot{\ell}_1 - m_1 g + \lambda = 0$$

$$\frac{d}{dt} m_2\dot{\ell}_2 - m_2 g + \lambda = 0$$

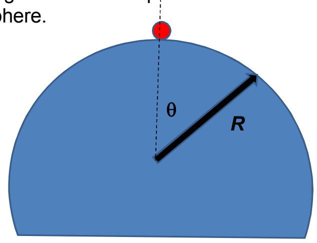
$$\dot{\ell}_1 + \dot{\ell}_2 = 0 = \ddot{\ell}_1 + \ddot{\ell}_2$$

$$\Rightarrow \lambda = \frac{2m_1 m_2}{m_1 + m_2} g$$

$$\ddot{\ell}_1 = -\ddot{\ell}_2 = \frac{m_1 - m_2}{m_1 + m_2} g$$

Another example:

A particle of mass *m* starts at rest on top of a smooth fixed hemisphere of radius *R*. Find the angle at which the particle leaves the hemisphere.



Example continued

Constraint Equation : $f(r,\theta) = r - R$

Lagrangian : $L(r,\theta,\dot{r},\dot{\theta}) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta$

Euler - Lagrangian equations :

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda\frac{\partial f}{\partial r} = 0 \qquad mr\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0 \qquad mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} = 0$$

Example continued

$$mr\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda = 0$$

$$mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} = 0$$

Using constraint :

$$mR\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$

$$mgR\sin\theta - mR^2\ddot{\theta} = 0$$

$$\ddot{\theta} = \frac{g}{R}\sin\theta \qquad \Rightarrow \dot{\theta}^2 = -\frac{2g}{R}\left(\cos\theta - 1\right)$$

$$\Rightarrow \lambda = mg(3\cos\theta - 2)$$