

PHY 711 Classical Mechanics and Mathematical Methods

10-10:50 AM MWF Olin 103

Plan for Lecture 11:

Continue reading Chapter 3 & 6

1. Constructing the Hamiltonian

2. Hamilton's canonical equation

3. Examples

9/18/2015

PHY 711 Fall 2015 -- Lecture 11

1

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/26/2015	Chap. 1	Review of basic principles	#1
2	Fri, 8/28/2015	Chap. 1	Scattering theory	#2
3	Mon, 8/31/2015	Chap. 1	Scattering theory continued	#3
4	Wed, 9/02/2015	Chap. 2	Accelerated coordinate systems	#4
5	Fri, 9/04/2015	Chap. 3	Calculus of variations	#5
6	Mon, 9/07/2015	Chap. 3	Calculus of variations	#6
7	Wed, 9/09/2015	Chap. 3	Hamilton's principle	#7
8	Fri, 9/11/2015	Chap. 3 & 6	Hamilton's principle	#8
9	Mon, 9/14/2015	Chap. 3 & 6	Lagrangians with constraints	#9
10	Wed, 9/16/2015	Chap. 3 & 6	Lagrangians and constants of motion	#10
11	Fri, 9/18/2015	Chap. 3 & 6	Hamiltonian formalism	#11
12	Mon, 9/21/2015	Chap. 3 & 6	Hamiltonian formalism	
13	Wed, 9/23/2015	Chap. 3 & 6	Hamiltonian Jacobi transformations	
14	Fri, 9/25/2015	Chap. 4	Small oscillations	
15	Mon, 9/28/2015	Chap. 4	Normal modes of motion	

9/18/2015

PHY 711 Fall 2015 -- Lecture 11

2

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

⇒ Second order differential equations for $q_\sigma(t)$

Switching variables – Legendre transformation

Define: $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \quad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

9/18/2015

PHY 711 Fall 2015 -- Lecture 11

3

Hamiltonian picture – continued

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \quad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$$

$$dH = \sum_\sigma \left(\dot{q}_\sigma dp_\sigma + p_\sigma d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma \right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma \right) + \frac{\partial H}{\partial t} dt$$

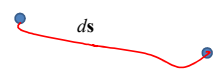
$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \quad \frac{\partial L}{\partial q_\sigma} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\sigma} \equiv \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

9/18/2015

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4

Direct application of Hamiltonian's principle using the Hamiltonian function --

Generalized coordinates:
 $q_\sigma(\{x_i\})$ Define -- Lagrangian: $L \equiv T - U$

$$L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)$$

$$\Rightarrow \text{Minimization integral: } S = \int_{t_i}^{t_f} L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) dt$$

Expressed in terms of Hamiltonian:

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$H = \sum_\sigma \dot{q}_\sigma p_\sigma - L \quad \Rightarrow L = \sum_\sigma \dot{q}_\sigma p_\sigma - H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

9/18/2015

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5

Hamilton's principle continued:

Minimization integral:

$$S = \int_{t_i}^{t_f} \left(\sum_\sigma \dot{q}_\sigma p_\sigma - H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t) \right) dt$$

$$\delta S = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\dot{q}_\sigma \delta p_\sigma + \delta \dot{q}_\sigma p_\sigma - \frac{\partial H}{\partial q_\sigma} \delta q_\sigma - \frac{\partial H}{\partial p_\sigma} \delta p_\sigma \right) \right) dt = 0$$

$$\Rightarrow \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$$

Canonical equations

$$\Rightarrow \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}$$

Detail:

$$\int_{t_i}^{t_f} \left(\sum_\sigma \delta \dot{q}_\sigma p_\sigma \right) dt = \int_{t_i}^{t_f} \left(\sum_\sigma \left(\frac{d(\delta q_\sigma p_\sigma)}{dt} - \delta q_\sigma \dot{p}_\sigma \right) \right) dt = \sum_\sigma \delta q_\sigma p_\sigma \Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} \left(\sum_\sigma \delta q_\sigma \dot{p}_\sigma \right) dt$$

9/18/2015

PHY 711 Fall 2015 – Lecture 11

6

Constants of the motion in Hamiltonian formalism

$$\begin{aligned}
 H &= H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t) \\
 \frac{dq_\sigma}{dt} &= \frac{\partial H}{\partial p_\sigma} \Rightarrow \text{constant } q_\sigma \text{ if } \frac{\partial H}{\partial p_\sigma} = 0 \\
 \frac{dp_\sigma}{dt} &= -\frac{\partial H}{\partial q_\sigma} \Rightarrow \text{constant } p_\sigma \text{ if } \frac{\partial H}{\partial q_\sigma} = 0 \\
 \frac{dH}{dt} &= \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial H}{\partial t} \\
 &\Rightarrow \text{constant } H \text{ if } \frac{\partial H}{\partial t} = 0
 \end{aligned}$$

9/18/2015

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7

Recipe for constructing the Hamiltonian and analyzing the equations of motion

1. Construct Lagrangian function : $L = L(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t)$
2. Compute generalized momenta : $p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$
3. Construct Hamiltonian expression : $H = \sum_\sigma \dot{q}_\sigma p_\sigma - L$
4. Form Hamiltonian function : $H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$
5. Analyze canonical equations of motion :

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \quad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

9/18/2015

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8

Example 1: one - dimensional potential :

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z)$$

$$p_x = m\dot{x} \quad p_y = m\dot{y} \quad p_z = m\dot{z}$$

$$H = m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2 - \left(\frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(z) \right)$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + V(z)$$

$$\text{Constants : } p_x, p_y, H$$

$$\text{Equations of motion : } \frac{dz}{dt} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \quad \frac{dp_z}{dt} = -\frac{dV}{dz}$$

9/18/2015

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9

Example 2: Motion in a central potential

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r)$$

$$p_r = m\dot{r} \quad p_\phi = mr^2\dot{\phi}$$

$$H = m\dot{r}^2 + mr^2\dot{\phi}^2 - \left(\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r) \right) \\ = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + V(r)$$

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r)$$

Constants : p_ϕ, H

Equations of motion :

$$\frac{dr}{dt} = \frac{p_r}{m} \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial r} = -\frac{p_\phi^2}{mr^3} - \frac{\partial V}{\partial r}$$

9/18/2015

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10

Other examples

Lagrangian for symmetric top with Euler angles α, β, γ :

$$L = L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 (\dot{\alpha} \cos \beta + \dot{\gamma})^2 - Mgh \cos \beta$$

$$p_\alpha = I_1 \dot{\alpha} \sin^2 \beta + I_3 (\dot{\alpha} \cos \beta + \dot{\gamma}) \cos \beta$$

$$p_\beta = I_1 \dot{\beta}$$

$$p_\gamma = I_3 (\dot{\alpha} \cos \beta + \dot{\gamma})$$

$$H = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 (\dot{\alpha} \cos \beta + \dot{\gamma})^2 + Mgh \cos \beta$$

$$H = \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + \frac{p_\beta^2}{2I_1} + \frac{p_\gamma^2}{2I_3} + Mgh \cos \beta$$

Constants : p_α, p_γ, H

9/18/2015

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11

Other examples

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{2c} B_0 (-\dot{x}y + y\dot{x})$$

$$p_x = m\dot{x} - \frac{q}{2c} B_0 y$$

$$p_y = m\dot{y} + \frac{q}{2c} B_0 x$$

$$p_z = m\dot{z}$$

$$H = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$H = \frac{\left(p_x + \frac{q}{2c} B_0 y \right)^2}{2m} + \frac{\left(p_y - \frac{q}{2c} B_0 x \right)^2}{2m} + \frac{p_z^2}{2m}$$

Constants : p_z, H

9/18/2015

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12

Canonical equations of motion for constant magnetic field:

$$H = \frac{\left(p_x + \frac{q}{2c} B_0 y\right)^2}{2m} + \frac{\left(p_y - \frac{q}{2c} B_0 x\right)^2}{2m} + \frac{p_z^2}{2m}$$

Constants : p_z, H

$$\frac{dx}{dt} = \frac{p_x + \frac{q}{2c} B_0 y}{m} \quad \frac{dy}{dt} = \frac{p_y - \frac{q}{2c} B_0 x}{m}$$

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial x} = -\frac{qB_0}{2mc} \left(p_y - \frac{q}{2c} B_0 x\right)$$

$$\frac{dp_y}{dt} = -\frac{\partial H}{\partial y} = -\frac{qB_0}{2mc} \left(p_x + \frac{q}{2c} B_0 y\right)$$

9/18/2015

PHY 711 Fall 2015 -- Lecture 11

13

Canonical equations of motion for constant magnetic field
-- continued:

$$\frac{dx}{dt} = \frac{p_x + \frac{q}{2c} B_0 y}{m} \quad \frac{dy}{dt} = \frac{p_y - \frac{q}{2c} B_0 x}{m}$$

$$\frac{dp_x}{dt} = \frac{qB_0}{2mc} \left(p_y - \frac{q}{2c} B_0 x\right) = \frac{qB_0}{2c} \frac{dy}{dt}$$

$$\frac{dp_y}{dt} = -\frac{qB_0}{2mc} \left(p_x + \frac{q}{2c} B_0 y\right) = -\frac{qB_0}{2c} \frac{dx}{dt}$$

$$\frac{d^2 x}{dt^2} = \frac{\dot{p}_x}{m} + \frac{q}{2mc} B_0 \dot{y} = \frac{qB_0}{mc} \frac{dy}{dt}$$

$$\frac{d^2 y}{dt^2} = \frac{\dot{p}_y}{m} - \frac{q}{2mc} B_0 \dot{x} = -\frac{qB_0}{mc} \frac{dx}{dt}$$

9/18/2015

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14

Poisson brackets:

Recall:

$$H = H(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \Rightarrow \text{constant } q_\sigma \text{ if } \frac{\partial H}{\partial p_\sigma} = 0$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma} \Rightarrow \text{constant } p_\sigma \text{ if } \frac{\partial H}{\partial q_\sigma} = 0$$

$$\frac{dH}{dt} = \sum_\sigma \left(\frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial H}{\partial t}$$

Similarly for an arbitrary function : $F = F(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$\frac{dF}{dt} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma} \dot{p}_\sigma \right) + \frac{\partial F}{\partial t} = \sum_\sigma \left(\frac{\partial F}{\partial q_\sigma} \frac{\partial H}{\partial p_\sigma} - \frac{\partial F}{\partial p_\sigma} \frac{\partial H}{\partial q_\sigma} \right) + \frac{\partial F}{\partial t}$$

9/18/2015

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15

Poisson brackets -- continued:

For an arbitrary function : $F = F(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t)$

$$\frac{dF}{dt} = \sum_{\sigma} \left(\frac{\partial F}{\partial q_{\sigma}} \dot{q}_{\sigma} + \frac{\partial F}{\partial p_{\sigma}} \dot{p}_{\sigma} \right) + \frac{\partial F}{\partial t} = \sum_{\sigma} \left(\frac{\partial F}{\partial q_{\sigma}} \frac{\partial H}{\partial p_{\sigma}} - \frac{\partial F}{\partial p_{\sigma}} \frac{\partial H}{\partial q_{\sigma}} \right) + \frac{\partial F}{\partial t}$$

Define :

$$[F, G]_{PB} \equiv \sum_{\sigma} \left(\frac{\partial F}{\partial q_{\sigma}} \frac{\partial G}{\partial p_{\sigma}} - \frac{\partial F}{\partial p_{\sigma}} \frac{\partial G}{\partial q_{\sigma}} \right) = -[G, F]_{PB}$$

So that : $\frac{dF}{dt} = [F, H]_{PB} + \frac{\partial F}{\partial t}$

9/18/2015

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16

Poisson brackets -- continued:

$$[F, G]_{PB} \equiv \sum_{\sigma} \left(\frac{\partial F}{\partial q_{\sigma}} \frac{\partial G}{\partial p_{\sigma}} - \frac{\partial F}{\partial p_{\sigma}} \frac{\partial G}{\partial q_{\sigma}} \right) = -[G, F]_{PB}$$

Examples :

$$[x, x]_{PB} = 0 \quad [x, p_x]_{PB} = 1 \quad [x, p_y]_{PB} = 0$$

Liouville theorem

Let $D \equiv$ density of particles in phase space :

$$\frac{dD}{dt} = [D, H]_{PB} + \frac{\partial D}{\partial t} = 0$$

9/18/2015

PHY 711 Fall 2015 -- Lecture 11

17