PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 13:

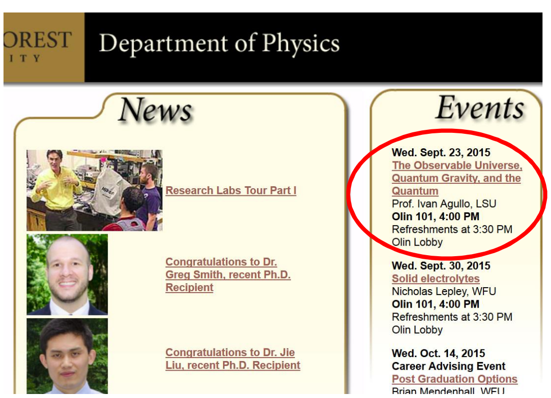
Finish reading Chapter 6

1. Virial theorem
2. Canonical transformations
3. Hamilton-Jacobi formalism

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/26/2015	Chap. 1	Review of basic principles	#1
2	Fri, 8/28/2015	Chap. 1	Scattering theory	#2
3	Mon, 8/31/2015	Chap. 1	Scattering theory continued	#3
4	Wed, 9/02/2015	Chap. 2	Accelerated coordinate systems	#4
5	Fri, 9/04/2015	Chap. 3	Calculus of variations	#5
6	Mon, 9/07/2015	Chap. 3	Calculus of variations	#6
7	Wed, 9/09/2015	Chap. 3	Hamilton's principle	#7
8	Fri, 9/11/2015	Chap. 3 & 6	Hamilton's principle	#8
9	Mon, 9/14/2015	Chap. 3 & 6	Lagrangians with constraints	#9
10	Wed, 9/16/2015	Chap. 3 & 6	Lagrangians and constants of motion	#10
11	Fri, 9/18/2015	Chap. 3 & 6	Hamiltonian formalism	#11
12	Mon, 9/21/2015	Chap. 3 & 6	Hamiltonian formalism	#12
13	Wed, 9/23/2015	Chap. 3 & 6	Hamiltonian Jacobi transformations	#13
14	Fri, 9/25/2015	Chap. 4	Small oscillations	
15	Mon, 9/28/2015	Chap. 4	Normal modes of motion	
16	Wed, 9/30/2015	Chap. 4	Normal modes of motion	
17	Fri, 10/02/2015	Chap. 4	Normal modes of motion	



WFU Physics Colloquium

TITLE: The Observable Universe, Gravity, and the Quantum

SPEAKER: Professor Ivan Agullo,

Department of Physics and Astronomy,
Louisiana State University

TIME: Wednesday September 23, 2015 at 4:00 PM

PLACE: Room 101 Olin Physical Laboratory

Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

ABSTRACT

An important difficulty in the search for a satisfactory theory of quantum gravity is the absence of experimental guidance. The astonishing improvement in cosmological observations attained in the last years offers an exciting opportunity to change this situation. It is believed that the anisotropies observed in the cosmic microwave background were originated in the very early universe. Observing their details could therefore tell us about physics in such extreme conditions.

In this talk, I will review the physics of the genesis of cosmic non-uniformities, paying special attention to the interplay between quantum effects and gravitation. I will describe how the forthcoming observations could provide detailed information about processes where the relationship between gravity and quantum mechanics plays a crucial role.

Virial theorem (Clausius ~ 1860)

$$2\langle T\rangle = -\left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r}_\sigma \right\rangle$$

Proof:

Define: $A \equiv \sum_\sigma \mathbf{p}_\sigma \cdot \mathbf{r}_\sigma$

$$\frac{dA}{dt} = \sum_\sigma \left(\dot{\mathbf{p}}_\sigma \cdot \mathbf{r}_\sigma + \mathbf{p}_\sigma \cdot \dot{\mathbf{r}}_\sigma \right) = \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r} + 2T$$

$$\left\langle \frac{dA}{dt} \right\rangle = \left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r} \right\rangle + 2\langle T\rangle$$

$$\left\langle \frac{dA}{dt} \right\rangle = \frac{1}{\tau}\int_0^\tau \frac{dA(t)}{dt}dt = \frac{A(\tau)-A(0)}{\tau} \Rightarrow 0$$

$$\Rightarrow \left\langle \sum_\sigma \mathbf{F}_\sigma \cdot \mathbf{r} \right\rangle + 2\langle T\rangle = 0$$

Hamiltonian formalism

$$H = H(\{q_\sigma(t)\},\{p_\sigma(t)\},t)$$

Canonical equations of motion

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma}$$

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

Notion of "Canonical" transformations

$$q_\sigma = q_\sigma\left(\{Q_1 \cdots Q_n\},\{P_1 \cdots P_n\},t\right) \qquad \text{for each } \sigma$$
$$p_\sigma = p_\sigma\left(\{Q_1 \cdots Q_n\},\{P_1 \cdots P_n\},t\right) \qquad \text{for each } \sigma$$

For some $\tilde{H}$ and F,

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H\left(\{q_\sigma\},\{p_\sigma\},t\right) = \sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) + \frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right)$$

Apply Hamilton's principle:

$$\delta \int_{t_i}^{t_f} \left[\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) + \frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right)\right] dt = 0$$

$$\delta \int_{t_i}^{t_f} \left[\frac{d}{dt}F\left(\{q_\sigma\},\{Q_\sigma\},t\right)\right] dt = \int_{t_i}^{t_f} \left[\frac{d}{dt}\delta F\left(\{q_\sigma\},\{Q_\sigma\},t\right)\right] dt$$

$$= \delta F\left(t_f\right) - \delta F\left(t_i\right) = 0 \qquad \text{and} \qquad \dot{Q}_\sigma = \frac{\partial \tilde{H}}{\partial P_\sigma} \qquad \dot{P}_\sigma = -\frac{\partial \tilde{H}}{\partial Q_\sigma}$$

9/23/2015 PHY 711 Fall 2015 -- Lecture 13 7

Some relations between old and new variables:

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$
$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t)$$

$$\frac{d}{dt}F(\{q_\sigma\},\{Q_\sigma\},t) = \sum_\sigma\left(\left(\frac{\partial F}{\partial q_\sigma}\right)\dot{q}_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\dot{Q}_\sigma\right) + \frac{\partial F}{\partial t}$$

$$\Rightarrow \sum_\sigma\left(p_\sigma - \left(\frac{\partial F}{\partial q_\sigma}\right)\right)\dot{q}_\sigma - H\left(\{q_\sigma\},\{p_\sigma\},t\right) =$$
$$\sum_\sigma\left(P_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\right)\dot{Q}_\sigma - \tilde{H}\left(\{Q_\sigma\},\{P_\sigma\},t\right) + \frac{\partial F}{\partial t}$$

9/23/2015 PHY 711 Fall 2015 -- Lecture 13 8

$$\sum_\sigma\left(p_\sigma - \left(\frac{\partial F}{\partial q_\sigma}\right)\right)\dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$
$$\sum_\sigma\left(P_\sigma + \left(\frac{\partial F}{\partial Q_\sigma}\right)\right)\dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{\partial F}{\partial t}$$

$$\Rightarrow p_\sigma = \left(\frac{\partial F}{\partial q_\sigma}\right) \qquad P_\sigma = -\left(\frac{\partial F}{\partial Q_\sigma}\right)$$

$$\Rightarrow \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) = H(\{q_\sigma\},\{p_\sigma\},t) + \frac{\partial F}{\partial t}$$

9/23/2015 PHY 711 Fall 2015 -- Lecture 13 9

Note that it is conceivable that if we were extraordinarily clever, we could find all of the constants of the motion!

$$\dot{Q}_\sigma = \frac{\partial \tilde{H}}{\partial P_\sigma} \qquad \dot{P}_\sigma = -\frac{\partial \tilde{H}}{\partial Q_\sigma}$$

$$\text{Suppose: } \dot{Q}_\sigma = \frac{\partial \tilde{H}}{\partial P_\sigma} = 0 \;\text{ and }\; \dot{P}_\sigma = -\frac{\partial \tilde{H}}{\partial Q_\sigma} = 0$$

$$\Rightarrow Q_\sigma, P_\sigma \quad \text{are constants of the motion}$$

Possible solution – Hamilton-Jacobi theory:

$$\text{Suppose: } \quad F(\{q_\sigma\},\{Q_\sigma\},t) \Rightarrow -\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)$$

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) + \frac{d}{dt}\left(-\sum_\sigma P_\sigma Q_\sigma + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

$$= -\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) - \sum_\sigma \dot{P}_\sigma Q_\sigma + \sum_\sigma \left(\frac{\partial S}{\partial q_\sigma}\dot{q}_\sigma + \frac{\partial S}{\partial P_\sigma}\dot{P}_\sigma\right) + \frac{\partial S}{\partial t}$$

Solution :

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \qquad Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$

$$\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t) = H(\{q_\sigma\},\{p_\sigma\},t) + \frac{\partial S}{\partial t}$$

When the dust clears :

Assume $\{Q_\sigma\},\{P_\sigma\},\tilde{H}$ are constants; choose $\tilde{H} = 0$

Need to find $S(\{q_\sigma\},\{P_\sigma\},t)$

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} \qquad Q_\sigma = \frac{\partial S}{\partial P_\sigma}$$

$$\Rightarrow H\left(\{q_\sigma\},\left\{\frac{\partial S}{\partial q_\sigma}\right\},t\right) + \frac{\partial S}{\partial t} = 0$$

Note : S is the "action":

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\sum_\sigma \cancelto{0}{P_\sigma \dot{Q}_\sigma} - \cancelto{0}{\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t)} + \frac{d}{dt}\left(-\sum_\sigma \cancelto{0}{P_\sigma Q_\sigma} + S(\{q_\sigma\},\{P_\sigma\},t)\right)$$

$$\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) =$$

$$\cancelto{0}{\sum_\sigma P_\sigma \dot{Q}_\sigma} - \cancelto{0}{\tilde{H}(\{Q_\sigma\},\{P_\sigma\},t)} + \frac{d}{dt}\left(-\cancelto{0}{\sum_\sigma P_\sigma Q_\sigma} + S(\{q_\sigma\},\{P_\sigma\},t) \right)$$

$$\int_{t_i}^{t_f} \left(\sum_\sigma p_\sigma \dot{q}_\sigma - H(\{q_\sigma\},\{p_\sigma\},t) \right) dt = \int_{t_i}^{t_f} \left(\frac{d}{dt}\left(S(\{q_\sigma\},\{P_\sigma\},t) \right) \right) dt$$

$$= S(\{q_\sigma\},\{P_\sigma\},t)\Big|_{t_i}^{t_f}$$

Differential equation for ***S***:

$$H\left(\{q_\sigma\}, \left\{ \frac{\partial S}{\partial q_\sigma} \right\}, t \right) + \frac{\partial S}{\partial t} = 0$$

$$\text{Example:} \qquad H(\{q\},\{p\},t) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

$$\text{Hamilton - Jacobi Eq:} \quad H\left(\{q\}, \left\{ \frac{\partial S}{\partial q} \right\}, t \right) + \frac{\partial S}{\partial t} = 0$$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q} \right)^2 + \frac{1}{2} m\omega^2 q^2 + \frac{\partial S}{\partial t} = 0$$

$$\text{Assume:} \quad S(q,t) \equiv W(q) - Et \qquad (E \quad \text{constant})$$

Continued:

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q} \right)^2 + \frac{1}{2} m\omega^2 q^2 + \frac{\partial S}{\partial t} = 0$$

$$\text{Assume:} \quad S(q,t) \equiv W(q) - Et \qquad (E \quad \text{constant})$$

$$\frac{1}{2m}\left(\frac{dW}{dq} \right)^2 + \frac{1}{2} m\omega^2 q^2 = E$$

$$\frac{dW}{dq} = \sqrt{2mE - (m\omega)^2 q^2}$$

$$W(q) = \int \sqrt{2mE - (m\omega)^2 q^2}\, dq$$

Continued:

$$W(q)=\int\sqrt{2mE-(m\omega)^2q^2}\,dq$$
$$=\frac{1}{2}q\sqrt{2mE-(m\omega)^2q^2}+\frac{E}{\omega}\sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}}\right)+C$$
$$S(q,E,t)=\frac{1}{2}q\sqrt{2mE-(m\omega)^2q^2}+\frac{E}{\omega}\sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}}\right)-Et$$
$$\frac{\partial S}{\partial E}=Q=\frac{1}{\omega}\sin^{-1}\left(\frac{m\omega q}{\sqrt{2mE}}\right)-t$$
$$\Rightarrow q(t)=\frac{\sqrt{2mE}}{m\omega}\sin(\omega(t+Q))$$

Another example of Hamilton Jacobi equations

Example: $H(\{q\},\{p\},t)=\dfrac{p^2}{2m}+mgy$

Assume $y(0)=h;\qquad p(0)=0$

Hamilton-Jacobi Eq: $H\left(\{q\},\left\{\dfrac{\partial S}{\partial q}\right\},t\right)+\dfrac{\partial S}{\partial t}=0$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2+mgy+\frac{\partial S}{\partial t}=0$$

Assume: $S(y,t)\equiv W(y)-Et\qquad(E\ \text{ constant})$

Example: $H(\{q\},\{p\},t)=\dfrac{p^2}{2m}+mgy$

Assume $y(0)=h;\qquad p(0)=0$

$$\frac{1}{2m}\left(\frac{dW}{dy}\right)^2+mgy=E\equiv mgh$$
$$W(y)=m\int_y^h\sqrt{2g(h-y')}\,dy'=\frac{2}{3}m\sqrt{2g}\,(h-y)^{3/2}$$
$$S(y,t)=W(y)-Et=\frac{2}{3}m\sqrt{2g}\,(h-y)^{3/2}-mght$$

Check action:

For this case: $y(t) = h - \frac{1}{2}gt^2$

$$S = \int_0^t \left(\frac{1}{2}m\dot{y}^2 - mgy \right) dt' = \frac{1}{3}mg^2t^3 - mght$$

$$S(y,t) = W(y) - Et = \frac{2}{3}m\sqrt{2g}\left(h-y\right)^{3/2} - mght$$

9/23/2015 PHY 711 Fall 2015 -- Lecture 13 19

Recap --

Lagrangian picture

For independent generalized coordinates $q_\sigma(t)$:

$$L = L\left(\{q_\sigma(t)\}, \{\dot{q}_\sigma(t)\}, t\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$\Rightarrow$ Second order differential equations for $q_\sigma(t)$

Hamiltonian picture

$$H = H\left(\{q_\sigma(t)\}, \{p_\sigma(t)\}, t\right)$$

$$\frac{dq_\sigma}{dt} = \frac{\partial H}{\partial p_\sigma} \qquad \frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q_\sigma}$$

$\Rightarrow$ Coupled first order differential equations for

$q_\sigma(t)$ and $p_\sigma(t)$

9/23/2015 PHY 711 Fall 2015 -- Lecture 13 20