# PHY 711 Classical Mechanics and Mathematical Methods
## 10-10:50 AM MWF Olin 103

**Plan for Lecture 17:**
**Appendix A from Fetter & Walecka**

**Review/presentation of some useful mathematical tools**

1. Complex variables
2. Contour integration
3. Fourier transforms



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|---|---|---|---|---|
| 4 | Wed, 9/02/2015 | Chap. 2 | Accelerated coordinate systems | #4 |
| 5 | Fri, 9/04/2015 | Chap. 3 | Calculus of variations | #5 |
| 6 | Mon, 9/07/2015 | Chap. 3 | Calculus of variations | #6 |
| 7 | Wed, 9/09/2015 | Chap. 3 | Hamilton's principle | #7 |
| 8 | Fri, 9/11/2015 | Chap. 3 & 6 | Hamilton's principle | #8 |
| 9 | Mon, 9/14/2015 | Chap. 3 & 6 | Lagrangians with constraints | #9 |
| 10 | Wed, 9/16/2015 | Chap. 3 & 6 | Lagrangians and constants of motion | #10 |
| 11 | Fri, 9/18/2015 | Chap. 3 & 6 | Hamiltonian formalism | #11 |
| 12 | Mon, 9/21/2015 | Chap. 3 & 6 | Hamiltonian formalism | #12 |
| 13 | Wed, 9/23/2015 | Chap. 3 & 6 | Hamiltonian Jacobi transformations | #13 |
| 14 | Fri, 9/25/2015 | Chap. 4 | Small oscillations | #14 |
| 15 | Mon, 9/28/2015 | Chap. 4 | Normal modes of motion | #15 |
| 16 | Wed, 9/30/2015 | Chap. 7 | Wave motion | #16 |
| 17 | Fri, 10/02/2015 | Chap. 7 & App. A | Contour Integration | #17 |
| 18 | Mon, 10/05/2015 | Chap. 7 | Wave motion | |
| 19 | Wed, 10/07/2015 | Chap. 7 | Sturm-Liouville Equations | |
| 20 | Fri, 10/09/2015 | Chap. 7 | Sturm-Liouville Equations | |
| | Mon, 10/12/2015 | | No class | Take home exam |
| | Wed, 10/14/2015 | | No class | Take home exam due |
| | Fri, 10/16/2015 | | Fall break -- no class | |



Complex numbers

$$i \equiv \sqrt{-1} \qquad i^2 = -1$$

Define $z = x + iy$

$$|z|^2 = zz^* = (x+iy)(x-iy) = x^2 + y^2$$

Polar representation

$$z = \rho(\cos\phi + i\sin\phi) = \rho\, e^{i\phi}$$

Functions of complex variables

$$f(z) = \Re(f(z)) + i\Im(f(z)) \equiv u(x,y) + iv(x,y)$$

Derivatives:

$$\frac{\partial f(z)}{\partial x} = \frac{\partial u(z)}{\partial x} + i\frac{\partial v(z)}{\partial x} \qquad \frac{\partial f(z)}{i\partial y} = \frac{\partial u(z)}{i\partial y} + i\frac{\partial v(z)}{i\partial y}$$

$$\Rightarrow \frac{\partial u(z)}{\partial x} = \frac{\partial v(z)}{\partial y} \qquad \frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y}$$

Cauchy-Riemann equations

Analytic function

$f(z)$ is analytic if it is:

- continuous
- single valued
- its first derivative satisfies Cauchy-Rieman conditions

Which of the following functions are analytic?

$f(z)=e^z$

$f(z)=z^n$

$f(z)=\ln z$

$f(z)=z^\alpha$



Some details

$\ln z=\ln\left(\rho e^{i\phi}\right)=\ln\rho+\ \ln e^{i\phi}=\ln\rho+i\phi$

can find analytic region for $f\left(z\right)=\ln z$ such as $\rho>0$ and $-\pi\leq\phi\leq\pi$



**PHY 711 – Contour Integration**

These notes summarize some basic properties of complex functions and their integrals. An *analytic* function $f(z)$ in a certain region of the complex plane $z$ is one which takes a single (non-infinite) value and is differentiable within that region. Cauchy's theorm states that a closed contour integral of the function within that region has the value

$$\oint_C f(z)=0. \quad (1)$$

As an example, functions composed of integer powers of $z$ –

$$f(z)=z^n, \quad \text{for} \quad n=0,1,\pm2,\pm3.... \quad (2)$$

fall in this catogory. Notice that non-integral powers are generally not analytic and that $n=-1$ is also special. In fact, we can show that

$$\oint_C \frac{dz}{z}=2\pi i. \quad (3)$$

$$\oint_C \frac{dz}{z} = 2\pi i. \tag{3}$$

This result follows from the fact that we can deform the contour to a unit circle about the origin so that $z = e^{i\theta}$. Then

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta}} i d\theta = 2\pi i. \tag{4}$$

One result of this analysis is the Cauchy integral formula which states that for any analytic function $f(z)$ within a region $C$,

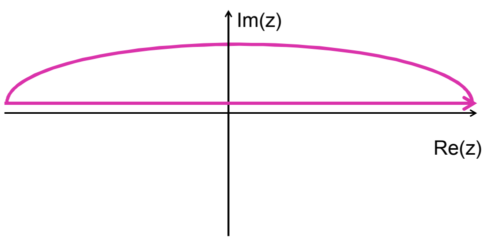
$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} dz'. \tag{5}$$



Example

Suppose $f(|z| \to \infty) = 0$ and for $z = x$:

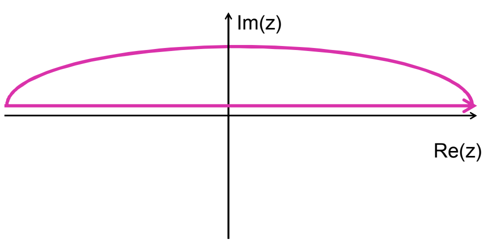
$$f(x) = a(x) + ib(x)$$





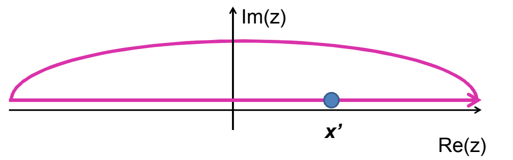

Example -- continued

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(z')}{z' - z} dz' \quad \text{where} \quad f(x) = a(x) + ib(x)$$




$$a(x) + ib(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{a(x') + ib(x')}{x' - x} dx'$$

Example -- continued




$$\int_{-\infty}^{\infty} \frac{f(x')}{x'-x} dx' = \int_{-\infty}^{x-\varepsilon} \frac{f(x')}{x'-x} dx' + \int_{x+\varepsilon}^{\infty} \frac{f(x')}{x'-x} dx' + \int_{x-\varepsilon}^{x+\varepsilon} \frac{f(x')}{x'-x} dx'$$

$$= P\int_{-\infty}^{\infty} \frac{f(x')}{x'-x} dx' + i\pi\, f(x)$$

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Example -- continued

$$\int_{-\infty}^{\infty} \frac{f(x')}{x'-x} dx' = \int_{-\infty}^{x-\varepsilon} \frac{f(x')}{x'-x} dx' + \int_{x+\varepsilon}^{\infty} \frac{f(x')}{x'-x} dx' + \int_{x-\varepsilon}^{x+\varepsilon} \frac{f(x')}{x'-x} dx'$$

$$= P\int_{-\infty}^{\infty} \frac{f(x')}{x'-x} dx' + i\pi\, f(x)$$

$$a(x) + ib(x) = \frac{P}{2\pi i}\int_{-\infty}^{\infty} \frac{a(x') + ib(x')}{x'-x} dx' + \frac{\pi i}{2\pi i}\left(a(x) + ib(x)\right)$$

$$\Rightarrow a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{b(x')}{x'-x} dx' \qquad b(x) = -\frac{P}{\pi}\int_{-\infty}^{\infty} \frac{a(x')}{x'-x} dx'$$

Kramers-Kronig relationship

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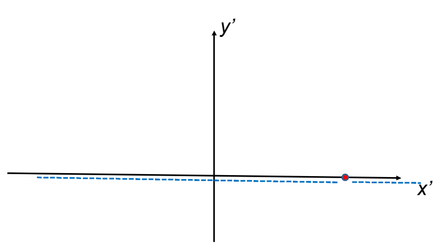
Comment on evaluating principal parts integrals

$$a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{b(x')}{x'-x} dx' = \lim_{\epsilon \to 0}\left( \frac{1}{\pi}\int_{-\infty}^{x-\epsilon} \frac{b(x')}{x'-x} dx' + \frac{1}{\pi}\int_{x+\epsilon}^{\infty} \frac{b(x')}{x'-x} dx' \right)$$

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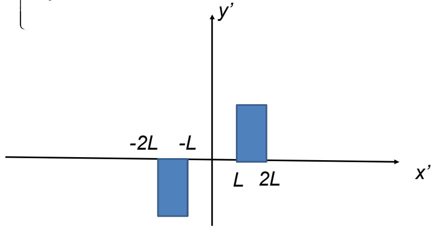
## Comment on evaluating principal parts integrals

Example:

$$b(x') = \begin{cases} 0 & \text{for } x' < -2L, \quad -L < x' < L, \quad x' > 2L \\ B_0 & \text{for } \; L < x' < 2L \\ -B_0 & \text{for } \; -2L < x' < -L \end{cases}$$






$$a(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{b(x')}{x'-x}dx' = \lim_{\epsilon \to 0}\left(\frac{1}{\pi}\int_{-\infty}^{x-\epsilon} \frac{b(x')}{x'-x}dx' + \frac{1}{\pi}\int_{x+\epsilon}^{\infty} \frac{b(x')}{x'-x}dx'\right)$$

For our example:

$$a(x) = \frac{B_0}{\pi}\ln\left(\left|\frac{4L^2 - x^2}{L^2 - x^2}\right|\right)$$



Another result of this analysis is the Residue Theorm which states that if the complex function $g(z)$ has poles at a finite number of points $z_p$ within a region $C$ but is otherwise analytic, the contour integral can be evaluated according to

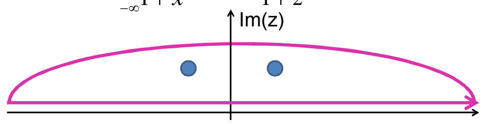
$$\oint_C g(z)dz = 2\pi i \sum_p Res(g_p), \tag{6}$$

where the residue is given by

$$Res(g_p) \equiv \lim_{z \to z_p}\left\{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left((z-z_p)^m g(z)\right)\right\}, \tag{7}$$

where $m$ denotes the order of the pole.

Example: $\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}\,dx = \oint \frac{z^2}{1+z^4}\,dz$




$$\oint \frac{z^2}{1+z^4}\,dz = 2\pi i\left(\mathrm{Res}\left(z_p = e^{i\pi/4}\right) + \mathrm{Res}\left(z_p = e^{3i\pi/4}\right)\right)$$

$$1+z^4 = \left(z-e^{i\pi/4}\right)\left(z-e^{3i\pi/4}\right)\left(z-e^{-i\pi/4}\right)\left(z-e^{-3i\pi/4}\right)$$

$$\mathrm{Res}\left(z_p = e^{i\pi/4}\right) = \frac{e^{i\pi/4}}{4i} \qquad \mathrm{Res}\left(z_p = e^{3i\pi/4}\right) = -\frac{e^{3i\pi/4}}{4i}$$

$$\oint \frac{z^2}{1+z^4}\,dz = 2\pi i\left(\frac{e^{i\pi/4}}{4i} - \frac{e^{3i\pi/4}}{4i}\right) = \frac{\pi}{\sqrt{2}}$$



Contour integral for homework:

$$\int_0^\infty \frac{\cos(ax)}{4x^4+5x^2+1}\,dx.$$



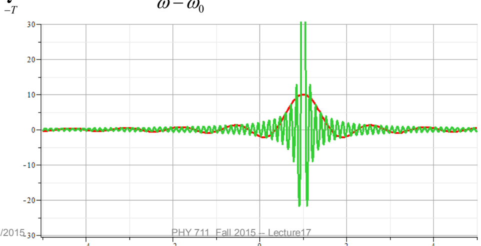
Fourier transforms

A useful identity

$$\int_{-\infty}^{\infty} dt\, e^{-i(\omega-\omega_0)t} = 2\pi\delta(\omega-\omega_0)$$

Note that

$$\int_{-T}^{T} dt\, e^{-i(\omega-\omega_0)t} = \frac{2\sin[(\omega-\omega_0)T]}{\omega-\omega_0}$$



Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega\, F(\omega)\, e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t)\, e^{i\omega t}$$

Check:

$$f(t) = \int_{-\infty}^{\infty} d\omega \left( \frac{1}{2\pi}\int_{-\infty}^{\infty} dt'\, f(t')\, e^{i\omega t'} \right) e^{-i\omega t}$$

$$f(t) = \int_{-\infty}^{\infty} dt'\, f(t') \left( \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t'-t)} \right) = \int_{-\infty}^{\infty} dt'\, f(t')\, \delta(t'-t)$$

Note: The location of the $2\pi$ factor varies among texts.



Properties of Fourier transforms -- Parseval's theorem:

$$\int_{-\infty}^{\infty} dt\, (f(t))^* f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, (F(\omega))^* F(\omega)$$

Check:

$$\int_{-\infty}^{\infty} dt\, (f(t))^* f(t) = \int_{-\infty}^{\infty} dt \left( \left( \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F(\omega) e^{i\omega t} \right)^* \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' F(\omega') e^{i\omega' t} \right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F^*(\omega) \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' F(\omega') \int_{-\infty}^{\infty} dt\, e^{i(\omega'-\omega)t}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F^*(\omega) \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' F(\omega')\, 2\pi\delta(\omega'-\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F^*(\omega) F(\omega)$$



## Use of Fourier transforms to solve wave equation

Wave equation: $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$

Suppose $u(x,t) = e^{-i\omega t}\tilde{F}(x,\omega)$ where $\tilde{F}(x,\omega)$ satisfies the equation:

$$\frac{\partial^2 \tilde{F}(x,\omega)}{\partial x^2} = -\frac{\omega^2}{c^2}\tilde{F}(x,\omega) \equiv -k^2 \tilde{F}(x,\omega)$$

Further assume that fixed boundary conditions apply: $0 \le x \le L$

with $\tilde{F}(0,\omega) = 0$ and $\tilde{F}(L,\omega) = 0$

For $n = 1,2,3\cdots$

$$\tilde{F}_n(x,\omega) = \sin\left(\frac{n\pi x}{L}\right) \qquad k \to k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$$

$$u(x,t) = e^{-i\omega_n t}\sin(k_n x) = e^{-i\omega_n t}\frac{\left(e^{ik_n x} - e^{-ik_n x}\right)}{2i} = \frac{\left(e^{ik_n(x-ct)} - e^{-ik_n(x+ct)}\right)}{2i}$$

Use of Fourier transforms to solve wave equation -- continued

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Using superposition : Suppose $u(x,t) = \sum_n C_n e^{-i\omega_n t} \widetilde{F}_n(x,\omega_n)$

$$\frac{\partial^2 \widetilde{F}_n(x,\omega_n)}{\partial x^2} = -\frac{\omega_n^2}{c^2} \widetilde{F}(x,\omega_n) \equiv -k_n^2 \widetilde{F}(x,\omega_n)$$

For $\widetilde{F}_n(x,\omega) = \sin\left(\frac{n\pi x}{L}\right)$ $\quad k \to k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$

$$\Rightarrow u(x,t) = \sum_n C_n e^{-i\omega_n t} \sin(k_n x) = \sum_n \frac{C_n}{2i} e^{-i\omega_n t}\left(e^{ik_n x} - e^{-ik_n x}\right)$$

$$= \sum_n \frac{C_n}{2i}\left(e^{ik_n(x-ct)} - e^{-ik_n(x+ct)}\right) \equiv f(x-ct) + g(x+ct)$$

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Fourier transform for periodic function :

Suppose $f(t+nT) = f(t)$ for and integer $n$

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t} = \sum_{n=-\infty}^{\infty}\left(\int_0^T dt\, f(t) e^{i\omega(t+nT)}\right)$$

Note that :

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega - \nu\Omega), \quad \text{where } \Omega \equiv \frac{2\pi}{T}$$

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Some details :

$$\sum_{n=-M}^{M} e^{in\omega T} = \frac{\sin\left(\left(M+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}$$

$$\lim_{M\to\infty}\left(\frac{\sin\left(\left(M+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}\right) = 2\pi\sum_\nu \delta(\omega T - \nu\Omega T) = \frac{2\pi}{T}\sum_\nu \delta(\omega - \nu\Omega)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega - \nu\Omega), \quad \text{where } \Omega \equiv \frac{2\pi}{T}$$

$$\Rightarrow F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t} = \sum_{\nu=-\infty}^{\infty} \Omega\delta(\omega - \nu\Omega)\left(\int_0^T dt\, f(t) e^{i\omega t}\right)$$

Thus, for a periodic function

$$f(t) = \sum_{\nu=-\infty}^{\infty} F(\nu\Omega) e^{-i\nu\Omega t}$$

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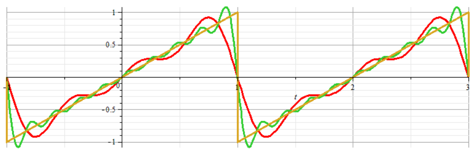
Example:

Suppose: $f(t)=\frac{t-nT}{T}$ for $(n-1)T\le t\le(n+1)T;\ n=0,2,4,6\ldots$

Note, in this case the repeat period is $2T$ and the convenient sample time interval is $-T\le t\le T$.

$$\bar{F}(\nu\Omega)=\frac{2\pi}{2T}i\int_{-T}^{T}\frac{t}{T}\sin\left(\frac{\nu\pi t}{2T}\right)dt \qquad f(t)=\sum_{\nu=1}^{\infty}2\left|\bar{F}(\nu\Omega)\right|\sin\left(\frac{\nu\pi t}{2T}\right)$$

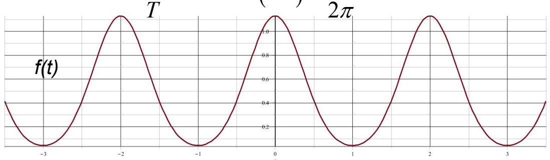


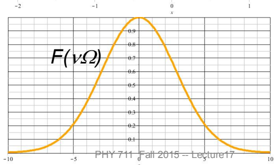


Example:

Suppose: $f(t)=\frac{1}{a\sqrt{\pi}}\sum_{n=-\infty}^{\infty}e^{-(t+nT)^2/a^2}=\sum_{\nu=-\infty}^{\infty}F(\nu\Omega)e^{-i\nu\Omega t}$

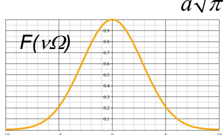
where $\Omega\equiv\frac{2\pi}{T}$ and $F(\nu\Omega)=\frac{1}{2\pi}e^{-a^2\nu^2\Omega^2/4}$



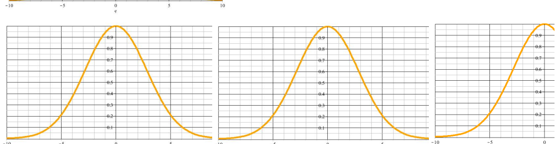






Continued: $f(t)=\frac{1}{a\sqrt{\pi}}\sum_{n=-\infty}^{\infty}e^{-(t+nT)^2/a^2}=\sum_{\nu=-\infty}^{\infty}F(\nu\Omega)e^{-i\nu\Omega t}$




Note: $f(t)\approx\sum_{\nu=-M}^{M}F(\nu\Omega)e^{-i\nu\Omega t}$




$$\Rightarrow f\left(\frac{mT}{2M+1}\right)=\sum_{\nu=-M}^{M}F(\nu\Omega)e^{-i2\pi\nu m/(2M+1)}$$

Thus, for a periodic function

$$f(t)=\sum_{\nu=-\infty}^{\infty}F(\nu\Omega)e^{-i\nu\Omega t}$$

Now suppose that the transformed function is bounded;

$$|F(\nu\Omega)|\leq\varepsilon \quad \text{for} \quad |\nu|\geq N$$

Define a periodic transform function function

$$\widetilde{F}(\nu\Omega)\equiv\widetilde{F}(\nu\Omega+\nu'((2N+1)\Omega))$$

Effect on time domain :

$$f(t)=\sum_{\nu=-\infty}^{\infty}\widetilde{F}(\nu\Omega)e^{-i\nu\Omega t} = \frac{2\pi}{(2N+1)\Omega}\sum_{\nu=-N}^{N}\widetilde{F}(\nu\Omega)e^{-i\nu\Omega t}\sum_{\mu}\delta\left(t-\frac{\mu T}{2N+1}\right)$$

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Doubly periodic functions

$$t\rightarrow\frac{\mu T}{2N+1}$$

$$\widetilde{f}_{\mu}=\frac{1}{2N+1}\sum_{\nu=-N}^{N}\widetilde{F}_{\nu}e^{-i2\pi\nu\mu/(2N+1)}$$

$$\widetilde{F}_{\nu}=\sum_{\mu=-N}^{N}\widetilde{f}_{\mu}e^{i2\pi\nu\mu/(2N+1)}$$

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More convenient notation

$$2N+1\rightarrow M$$

$$\widetilde{f}_{\mu}=\frac{1}{M}\sum_{\nu=0}^{M-1}\widetilde{F}_{\nu}e^{-i2\pi\nu\mu/M}$$

$$\widetilde{F}_{\nu}=\sum_{\mu=0}^{M}\widetilde{f}_{\mu}e^{i2\pi\nu\mu/M}$$

Note that for $W=e^{i2\pi/M}$

$$\widetilde{F}_0=\widetilde{f}_0W^0+\widetilde{f}_1W^0+\widetilde{f}_2W^0+\widetilde{f}_3W^0+\cdots$$

$$\widetilde{F}_1=\widetilde{f}_0W^0+\widetilde{f}_1W^1+\widetilde{f}_2W^2+\widetilde{f}_3W^3+\cdots$$

$$\widetilde{F}_2=\widetilde{f}_0W^0+\widetilde{f}_1W^2+\widetilde{f}_2W^4+\widetilde{f}_3W^6+\cdots$$

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Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \cdots$$
$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \cdots$$
$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \cdots$$

However, $W^M = \left(e^{i2\pi/M}\right)^M = 1$

and $W^{M/2} = \left(e^{i2\pi/M}\right)^{M/2} = -1$

Cooley-Tukey algorithm: J. W. Cooley and J. W. Tukey, "An algorithm for machine calculation of complex Fourier series" Math. Computation 19, 297-301 (1965)

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