PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 18:
Chapter 7 in Fetter & Walecka

Review/presentation of some useful mathematical tools

1. **Fourier series and transforms**
2. **Orthogonal function expansions**

3	Mon, 8/31/2015	Chap. 1	Scattering theory continued	#3
4	Wed, 9/02/2015	Chap. 2	Accelerated coordinate systems	#4
5	Fri, 9/04/2015	Chap. 3	Calculus of variations	#5
6	Mon, 9/07/2015	Chap. 3	Calculus of variations	#6
7	Wed, 9/09/2015	Chap. 3	Hamilton's principle	#7
8	Fri, 9/11/2015	Chap. 3 & 6	Hamilton's principle	#8
9	Mon, 9/14/2015	Chap. 3 & 6	Lagrangians with constraints	#9
10	Wed, 9/16/2015	Chap. 3 & 6	Lagrangians and constants of motion	#10
11	Fri, 9/18/2015	Chap. 3 & 6	Hamiltonian formalism	#11
12	Mon, 9/21/2015	Chap. 3 & 6	Hamiltonian formalism	#12
13	Wed, 9/23/2015	Chap. 3 & 6	Hamiltonian Jacobi transformations	#13
14	Fri, 9/25/2015	Chap. 4	Small oscillations	#14
15	Mon, 9/28/2015	Chap. 4	Normal modes of motion	#15
16	Wed, 9/30/2015	Chap. 7	Wave motion	#16
17	Fri, 10/02/2015	Chap. 7 & App. A	Contour Integration	#17
18	Mon, 10/05/2015	Chap. 7	Fourier transforms	#18
19	Wed, 10/07/2015	Chap. 7	Laplace transforms	#19
20	Fri, 10/09/2015	Chap. 7	Green's functions	Start exam
	Mon, 10/12/2015		No class	Take home exam
	Wed, 10/14/2015		No class	Exam due before 10/19/2015
	Fri, 10/16/2015		Fall break -- no class	

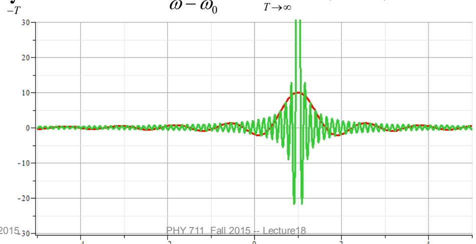
Fourier transforms

A useful identity

$$\int_{-\infty}^{\infty} dt\ e^{-i(\omega-\omega_0)t} = 2\pi\delta(\omega-\omega_0)$$

Note that

$$\int_{-T}^{T} dt\ e^{-i(\omega-\omega_0)t} = \frac{2\sin\left[(\omega-\omega_0)T\right]}{\omega-\omega_0} \underset{T\to\infty}{\approx} 2\pi\delta(\omega-\omega_0)$$



Definition of Fourier Transform for a function $f(t)$:

$$f(t) = \int_{-\infty}^{\infty} d\omega\, F(\omega)\, e^{-i\omega t}$$

Backward transform:

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t)\, e^{i\omega t}$$

Check:

$$f(t) = \int_{-\infty}^{\infty} d\omega \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dt'\, f(t')\, e^{i\omega t'} \right) e^{-i\omega t}$$

$$f(t) = \int_{-\infty}^{\infty} dt'\, f(t') \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t'-t)} \right) = \int_{-\infty}^{\infty} dt'\, f(t')\, \delta(t'-t)$$

Note: The location of the 2π factor varies among texts.

Properties of Fourier transforms -- Parseval's theorem:

$$\int_{-\infty}^{\infty} dt\, \left(f(t)\right)^* f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \left(F(\omega)\right)^* F(\omega)$$

Check:
$$\int_{-\infty}^{\infty} dt\, \left(f(t)\right)^* f(t) = \int_{-\infty}^{\infty} dt \left(\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F(\omega) e^{i\omega t} \right)^* \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' F(\omega') e^{i\omega' t} \right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F^*(\omega) \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' F(\omega') \int_{-\infty}^{\infty} dt\, e^{i(\omega'-\omega)t}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F^*(\omega) \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' F(\omega')\, 2\pi\delta(\omega'-\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F^*(\omega) F(\omega)$$

Use of Fourier transforms to solve wave equation

Wave equation: $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$

Suppose $u(x,t) = e^{-i\omega t}\widetilde{F}(x,\omega)$ where $\widetilde{F}(x,\omega)$ satisfies the equation:

$$\frac{\partial^2 \widetilde{F}(x,\omega)}{\partial x^2} = -\frac{\omega^2}{c^2}\widetilde{F}(x,\omega) \equiv -k^2 \widetilde{F}(x,\omega)$$

More generally:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F(x,\omega) e^{-i\omega t}$$

Further assume that fixed boundary conditions apply: $0 \le x \le L$

with $\widetilde{F}(0,\omega) = 0$ and $\widetilde{F}(L,\omega) = 0$

For $n = 1,2,3\cdots$

$$\widetilde{F}_n(x,\omega) = \sin\left(\frac{n\pi x}{L}\right) \qquad k \to k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$$

$$u(x,t) = e^{-i\omega_n t}\sin(k_n x) = e^{-i\omega_n t}\frac{\left(e^{ik_n x} - e^{-ik_n x}\right)}{2i} = \frac{\left(e^{ik_n(x-ct)} - e^{-ik_n(x+ct)}\right)}{2i}$$

Use of Fourier transforms to solve wave equation -- continued

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Using superposition : Suppose $u(x,t) = \sum_n C_n e^{-i\omega_n t} \widetilde{F}_n(x,\omega_n)$

$$\frac{\partial^2 \widetilde{F}_n(x,\omega_n)}{\partial x^2} = -\frac{\omega_n^2}{c^2} \widetilde{F}(x,\omega_n) \equiv -k_n^2 \widetilde{F}(x,\omega_n)$$

For $\widetilde{F}_n(x,\omega) = \sin\left(\frac{n\pi x}{L}\right)$ $\qquad k \to k_n = \frac{n\pi}{L} \equiv \frac{\omega_n}{c}$

$$\Rightarrow u(x,t) = \sum_n C_n e^{-i\omega_n t} \sin(k_n x) = \sum_n \frac{C_n}{2i} e^{-i\omega_n t}\left(e^{ik_n x} - e^{-ik_n x}\right)$$

$$= \sum_n \frac{C_n}{2i}\left(e^{ik_n(x-ct)} - e^{-ik_n(x+ct)}\right) \equiv f(x-ct) + g(x+ct)$$

Fourier transform for a time periodic function:

Suppose $f(t+nT) = f(t)$ for and integer n

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ f(t) e^{i\omega t} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_0^T dt\ f(t) e^{i\omega(t+nT)}\right)$$

Note that :

$$\sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega - \nu\Omega), \quad \text{where } \Omega \equiv \frac{2\pi}{T}$$

Some details :

$$\sum_{n=-M}^{M} e^{in\omega T} = \frac{\sin\left(\left(M+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}$$

$$\lim_{M\to\infty}\left(\frac{\sin\left(\left(M+\frac{1}{2}\right)\omega T\right)}{\sin\left(\frac{1}{2}\omega T\right)}\right) = 2\pi\sum_\nu \delta(\omega T - \nu\Omega T) = \frac{2\pi}{T}\sum_\nu \delta(\omega - \nu\Omega)$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} e^{in\omega T} = \Omega \sum_{\nu=-\infty}^{\infty} \delta(\omega - \nu\Omega), \quad \text{where } \Omega \equiv \frac{2\pi}{T}$$

$$\Rightarrow F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t) e^{i\omega t} = \sum_{\nu=-\infty}^{\infty} \Omega\delta(\omega - \nu\Omega)\left(\int_0^T dt\, f(t) e^{i\omega t}\right)$$

Thus, for a time periodic function

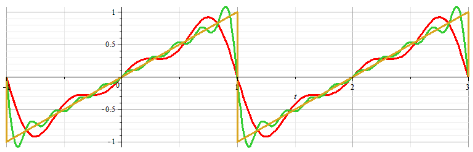
$$f(t) = \frac{1}{2\pi}\sum_{\nu=-\infty}^{\infty} F(\nu\Omega) e^{-i\nu\Omega t}$$

Example:

Suppose: $f(t)=\begin{cases}\dfrac{t-nT}{T} & \text{for } (n-1)T \le t \le (n+1)T;\ \ n=0,2,4,6\ldots \\ 0 & \text{otherwise}\end{cases}$

Note, in this case the repeat period is $2T$ and the convenient sample time interval is $-T \le t \le T$.

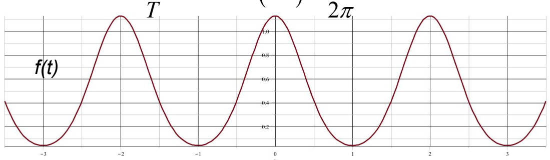
$$\bar{F}(\nu\Omega)=\frac{1}{2T}i\int_{-T}^{T}\frac{t}{T}\sin\left(\frac{\nu 2\pi t}{2T}\right)dt \qquad f(t)=\sum_{\nu=1}^{\infty}2\left|\bar{F}(\nu\Omega)\right|\sin\left(\frac{\nu 2\pi t}{2T}\right)$$

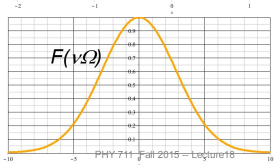


Example:

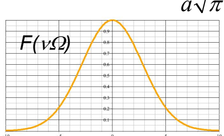
Suppose: $f(t)=\dfrac{1}{a\sqrt{\pi}}\displaystyle\sum_{n=-\infty}^{\infty}e^{-(t+nT)^2/a^2}=\sum_{\nu=-\infty}^{\infty}F(\nu\Omega)e^{-i\nu\Omega t}$

where $\Omega\equiv\dfrac{2\pi}{T}$ and $F(\nu\Omega)=\dfrac{1}{2\pi}e^{-a^2\nu^2\Omega^2/4}$

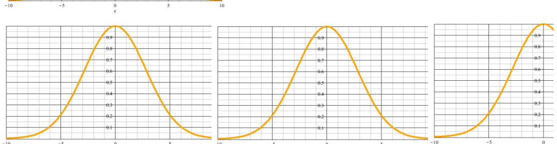




Continued: $f(t)=\dfrac{1}{a\sqrt{\pi}}\displaystyle\sum_{n=-\infty}^{\infty}e^{-(t+nT)^2/a^2}=\sum_{\nu=-\infty}^{\infty}F(\nu\Omega)e^{-i\nu\Omega t}$



Note: $f(t)\approx=\displaystyle\sum_{\nu=-M}^{M}F(\nu\Omega)e^{-i\nu\Omega t}$



ν=-M ν=M

$$\Rightarrow f\left(\frac{mT}{2M+1}\right)=\sum_{\nu=-M}^{M}F(\nu\Omega)e^{-i2\pi\nu m/(2M+1)}$$

Thus, for a periodic function

$$f(t)=\sum_{\nu=-\infty}^{\infty}F(\nu\Omega)e^{-i\nu\Omega t}$$

Now suppose that the transformed function is bounded;

$$|F(\nu\Omega)|\leq\varepsilon \quad \text{for} \quad |\nu|\geq N$$

Define a periodic transform function function

$$\widetilde{F}(\nu\Omega)\equiv\widetilde{F}(\nu\Omega+\nu'((2N+1)\Omega))$$

Effect on time domain :

$$f(t)=\sum_{\nu=-\infty}^{\infty}\widetilde{F}(\nu\Omega)e^{-i\nu\Omega t}=\frac{2\pi}{(2N+1)\Omega}\sum_{\nu=-N}^{N}\widetilde{F}(\nu\Omega)e^{-i\nu\Omega t}\sum_{\mu}\delta\left(t-\frac{\mu T}{2N+1}\right)$$

Doubly periodic functions

$$t\rightarrow\frac{\mu T}{2N+1}$$

$$\widetilde{f}_\mu=\frac{1}{2N+1}\sum_{\nu=-N}^{N}\widetilde{F}_\nu e^{-i2\pi\nu\mu/(2N+1)}$$

$$\widetilde{F}_\nu=\sum_{\mu=-N}^{N}\widetilde{f}_\mu e^{i2\pi\nu\mu/(2N+1)}$$

More convenient notation

$$2N+1\rightarrow M$$

$$\widetilde{f}_\mu=\frac{1}{M}\sum_{\nu=0}^{M-1}\widetilde{F}_\nu e^{-i2\pi\nu\mu/M}$$

$$\widetilde{F}_\nu=\sum_{\mu=0}^{M}\widetilde{f}_\mu e^{i2\pi\nu\mu/M}$$

Note that for $W=e^{i2\pi/M}$

$$\widetilde{F}_0=\widetilde{f}_0W^0+\widetilde{f}_1W^0+\widetilde{f}_2W^0+\widetilde{f}_3W^0+\cdots$$

$$\widetilde{F}_1=\widetilde{f}_0W^0+\widetilde{f}_1W^1+\widetilde{f}_2W^2+\widetilde{f}_3W^3+\cdots$$

$$\widetilde{F}_2=\widetilde{f}_0W^0+\widetilde{f}_1W^2+\widetilde{f}_2W^4+\widetilde{f}_3W^6+\cdots$$

Note that for $W = e^{i2\pi/M}$

$$\tilde{F}_0 = \tilde{f}_0 W^0 + \tilde{f}_1 W^0 + \tilde{f}_2 W^0 + \tilde{f}_3 W^0 + \cdots$$
$$\tilde{F}_1 = \tilde{f}_0 W^0 + \tilde{f}_1 W^1 + \tilde{f}_2 W^2 + \tilde{f}_3 W^3 + \cdots$$
$$\tilde{F}_2 = \tilde{f}_0 W^0 + \tilde{f}_1 W^2 + \tilde{f}_2 W^4 + \tilde{f}_3 W^6 + \cdots$$

However, $W^M = \left(e^{i2\pi/M}\right)^M = 1$

and $W^{M/2} = \left(e^{i2\pi/M}\right)^{M/2} = -1$

Cooley-Tukey algorithm: J. W. Cooley and J. W. Tukey, “An algorithm for machine calculation of complex Fourier series” Math. Computation 19, 297-301 (1965)