

**PHY 711 Classical Mechanics and
Mathematical Methods**
10-10:50 AM MWF Olin 103

Plan for Lecture 19:

**Introduction/review of mathematical
methods**

- 1. Laplace transform methods**
- 2. Green's function methods**

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3	Mon, 8/31/2015	Chap. 1	Scattering theory continued	#3
4	Wed, 9/02/2015	Chap. 2	Accelerated coordinate systems	#4
5	Fri, 9/04/2015	Chap. 3	Calculus of variations	#5
6	Mon, 9/07/2015	Chap. 3	Calculus of variations	#6
7	Wed, 9/09/2015	Chap. 3	Hamilton's principle	#7
8	Fri, 9/11/2015	Chap. 3 & 6	Hamilton's principle	#8
9	Mon, 9/14/2015	Chap. 3 & 6	Lagrangians with constraints	#9
10	Wed, 9/16/2015	Chap. 3 & 6	Lagrangians and constants of motion	#10
11	Fri, 9/18/2015	Chap. 3 & 6	Hamiltonian formalism	#11
12	Mon, 9/21/2015	Chap. 3 & 6	Hamiltonian formalism	#12
13	Wed, 9/23/2015	Chap. 3 & 6	Hamiltonian Jacobi transformations	#13
14	Fri, 9/25/2015	Chap. 4	Small oscillations	#14
15	Mon, 9/28/2015	Chap. 4	Normal modes of motion	#15
16	Wed, 9/30/2015	Chap. 7	Wave motion	#16
17	Fri, 10/02/2015	Chap. 7 & App. A	Contour Integration	#17
18	Mon, 10/05/2015	Chap. 7	Fourier transforms	#18
19	Wed, 10/07/2015	Chap. 7	Laplace transforms	#19
20	Fri, 10/09/2015	Chap. 7	Green's functions	Start exam
	Mon, 10/12/2015		No class	Take home exam
	Wed, 10/14/2015		No class	Exam due before 10/19/2015
	Fri, 10/16/2015		Fall break -- no class	

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Department of Physics

News



Research Labs Tour Part I



Congratulations to Dr. Greg Smith, recent Ph.D. Recipient



Congratulations to Dr. Jie Liu, recent Ph.D. Recipient

Events

Wed. Oct. 7, 2015
WFU Physics Research III
Theoretical/Computational
Olin 101, 4:00 PM
Refreshments at 3:30 PM
Olin Lobby

Wed. Oct. 14, 2015
Career Advising Event
Post Graduation Options
Brian Mendenhall, WFU
Salem 10 at 5:00 PM

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Laplace transforms

Laplace transforms can be used to solve initial value problems. The Laplace transform of a function $\phi(x)$ is defined as

$$\mathcal{L}_\phi(p) \equiv \int_0^\infty e^{-px} \phi(x) dx. \quad (24)$$

Assuming that $\phi(x)$ is well-behaved in the interval $0 \leq x < \infty$, the following properties are useful:

$$\mathcal{L}_{d\phi/dx}(p) = -\phi(0) + p\mathcal{L}_\phi(p), \quad (25)$$

and

$$\mathcal{L}_{d^2\phi/dx^2}(p) = -\frac{d\phi(0)}{dx} - p\phi(0) + p^2\mathcal{L}_\phi(p). \quad (26)$$

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These identities allow us to turn a differential equation for $\phi(x)$ into an algebraic equation for $\mathcal{L}_\phi(p)$. We then need to perform an inverse Laplace transform to find $\phi(x)$. For illustration, we will consider a simple example with $\tau(x) = 1$, $\sigma(x) = 1$, $\lambda = 0$. The differential equation then becomes

$$-\frac{d^2\phi(x)}{dx^2} = F(x), \quad (27)$$

where we will take the initial conditions to be $\phi(0) = 0$ and $d\phi(0)/dx = 0$. For our example, we will also take $F(x) = F_0 e^{-\gamma x}$. Multiplying both sides of the equation by e^{-px} and integrating $0 \leq x < \infty$, we find

$$\mathcal{L}_\phi(p) = -\frac{F_0}{p^2(\gamma + p)}. \quad (28)$$

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In general the inverse Laplace transform involves performing a contour integral, but we can use the following simple relations

$$\mathcal{L}_1 = \int_0^\infty e^{-px} dx = \frac{1}{p}, \quad (29)$$

$$\mathcal{L}_x = \int_0^\infty x e^{-px} dx = \frac{1}{p^2}, \quad (30)$$

$$\mathcal{L}_{e^{-\gamma x}} = \int_0^\infty e^{-\gamma x} e^{-px} dx = \frac{1}{p + \gamma}. \quad (31)$$

Noting that

$$-\frac{F_0}{p^2(\gamma + p)} = -\frac{F_0}{\gamma^2} \left(\frac{1}{\gamma + p} - \frac{1}{p} + \frac{\gamma}{p^2} \right), \quad (32)$$

we see that the inverse Laplace transform gives us

$$\phi(x) = \frac{F_0}{\gamma^2} (1 - e^{-\gamma x} - \gamma x). \quad (33)$$

We can check that this is a solution to the differential equation

$$-\frac{d^2\phi}{dx^2} = F_0 e^{-\gamma x} \quad \text{for} \quad \phi(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(0) = 0$$

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Using Laplace transforms to solve equation :

$$\left(-\frac{d^2}{dx^2} - 1\right)\phi(x) = F_0 \sin\left(\frac{\pi x}{L}\right) \quad \text{with } \phi(0) = 0, \frac{d\phi(0)}{dx} = 0$$

$$\mathcal{L}_\phi(p) = -\left(\frac{\pi}{L}\right) \frac{F_0}{\left(p^2 + 1\right)\left(p^2 + \left(\frac{\pi}{L}\right)^2\right)}$$

$$= -F_0 \left(\frac{\pi/L}{(\pi/L)^2 - 1}\right) \left(\frac{1}{p^2 + 1} - \frac{1}{p^2 + \left(\frac{\pi}{L}\right)^2}\right)$$

Note that : $\int_0^\infty \sin(at) e^{-pt} dt = \frac{a}{a^2 + p^2}$

$$\Rightarrow \phi(x) = \frac{F_0}{(\pi/L)^2 - 1} \left(\sin\left(\frac{\pi x}{L}\right) - \frac{\pi}{L} \sin(x)\right)$$

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Inverse Laplace transform :

$$\mathcal{L}_\phi(p) = \int_0^\infty e^{-pt} \phi(t) dt$$

$$\phi(t) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} e^{pt} \mathcal{L}_\phi(p) dp$$

Check : $\frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} e^{pt} \mathcal{L}_\phi(p) dp = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} e^{pt} dp \int_0^\infty e^{-pu} \phi(u) du$

$$\begin{aligned} \frac{1}{2\pi i} \int_0^\infty \phi(u) du \int_{\lambda - i\infty}^{\lambda + i\infty} e^{p(t-u)} dp &= \frac{1}{2\pi i} \int_0^\infty \phi(u) du \int_{-\infty}^\infty e^{\lambda(t-u)} e^{is(t-u)} ds \\ &= \frac{1}{2\pi i} \int_0^\infty \phi(u) du \left(e^{\lambda(t-u)} 2\pi i \delta(t-u) \right) \\ &= \begin{cases} \phi(t) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

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PHY 711 – Assignment #19

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Continue reading Chapter 7 in Fetter and Walecka.

1. Use the Laplace transform method to solve the following differential equation for $\phi(x)$ with initial conditions $\phi(0) = 0$ and $d\phi(0)/dx = 0$.

$$\left(-\frac{d^2}{dx^2} - 4\right)\phi(x) = F_0,$$

where F_0 is a given constant.

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Linear second-order ordinary differential equations Sturm-Liouville equations

Inhomogeneous problem: $\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \phi(x) = F(x)$

given functions

solution to be determined

applied force

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Solution methods of Sturm-Liouville equations (assume all functions and constants are real):

Homogeneous problem: $\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \phi_0(x) = 0$

Inhomogeneous problem: $\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \phi(x) = F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) \right) f_n(x) = \lambda_n \sigma(x) f_n(x)$$

It is possible to prove several properties of the eigenfunctions $f_n(x)$ – including orthogonality and completeness.

Orthogonality statement: $\int_a^b \sigma(x) f_n(x) f_m(x) dx = \delta_{nm} N_n$,

where $N_n \equiv \int_a^b \sigma(x) (f_n(x))^2 dx$.

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Completeness of eigenfunctions:

$$\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x')$$

Suppose that we can find a Green's function defined as follows:

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_\lambda(x, x') = \delta(x - x')$$

In terms of eigenfunctions:

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_\lambda(x, x') = \sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n}$$

$$\Rightarrow G_\lambda(x, x') = \sum_n \frac{f_n(x) f_n(x') / N_n}{\lambda_n - \lambda}$$

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Solution to inhomogeneous problem by using Green's functions

Inhomogeneous problem:

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) \phi(x) = F(x)$$

Green's function :

$$\left(-\frac{d}{dx} \tau(x) \frac{d}{dx} + v(x) - \lambda \sigma(x) \right) G_\lambda(x, x') = \delta(x - x')$$

Formal solution:

$$\phi_\lambda(x) = \phi_{\lambda 0}(x) + \int_0^L G_\lambda(x, x') F(x') dx'$$

Solution to homogeneous problem

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Example Sturm-Liouville problem:

Example: $\tau(x) = 1$; $\sigma(x) = 1$; $v(x) = 0$; $a = 0$ and $b = L$

$$\lambda = 1; \quad F(x) = F_0 \sin\left(\frac{\pi x}{L}\right)$$

Inhomogeneous equation :

$$\left(-\frac{d^2}{dx^2} - 1 \right) \phi(x) = F_0 \sin\left(\frac{\pi x}{L}\right)$$

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Eigenvalue equation :

$$\left(-\frac{d^2}{dx^2} \right) f_n(x) = \lambda_n f_n(x)$$

Eigenfunctions

$$f_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Eigenvalues :

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Completeness of eigenfunctions :

$$\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x')$$

In this example :

$$\frac{2}{L} \sum_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x'}{L}\right) = \delta(x - x')$$

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Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx} + v(x) - \lambda\sigma(x)\right)G_\lambda(x, x') = \delta(x - x')$$

Green's function for the example :

$$G(x, x') = \sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n - \lambda} = \frac{2}{L} \sum_n \frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n\pi}{L}\right)^2 - 1}$$

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Using Green's function to solve inhomogenous equation :

$$\left(-\frac{d^2}{dx^2} - 1\right)\phi(x) = F_0 \sin\left(\frac{\pi x}{L}\right)$$

$$\phi(x) = \phi_0(x) + \int_0^L G(x, x') F_0 \sin\left(\frac{\pi x'}{L}\right) dx'$$

$$= \phi_0(x) + \frac{2}{L} \sum_n \left[\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2 - 1} \int_0^L \sin\left(\frac{n\pi x'}{L}\right) F_0 \sin\left(\frac{\pi x'}{L}\right) dx' \right]$$

$$= \phi_0(x) + \frac{F_0}{\left(\frac{\pi}{L}\right)^2 - 1} \sin\left(\frac{\pi x}{L}\right)$$

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Alternate Green's function method :

$$G(x, x') = \frac{1}{W} g_a(x_<) g_b(x_>)$$

$$\left(-\frac{d^2}{dx^2} - 1\right)g_i(x) = 0 \quad \Rightarrow g_a(x) = \sin(x); \quad g_b(x) = \sin(L - x);$$

$$W = g_b(x) \frac{dg_a(x)}{dx} - g_a(x) \frac{dg_b(x)}{dx} = \sin(L - x) \cos(x) + \sin(x) \cos(L - x) = \sin(L)$$

$$\phi(x) = \phi_0(x) + \frac{\sin(L - x)}{\sin(L)} \int_0^x \sin(x') F_0 \sin\left(\frac{\pi x'}{L}\right) dx' + \frac{\sin(x)}{\sin(L)} \int_x^L \sin(L - x') F_0 \sin\left(\frac{\pi x'}{L}\right) dx'$$

$$\phi(x) = \phi_0(x) + \frac{F_0}{\left(\frac{\pi}{L}\right)^2 - 1} \sin\left(\frac{\pi x}{L}\right)$$

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