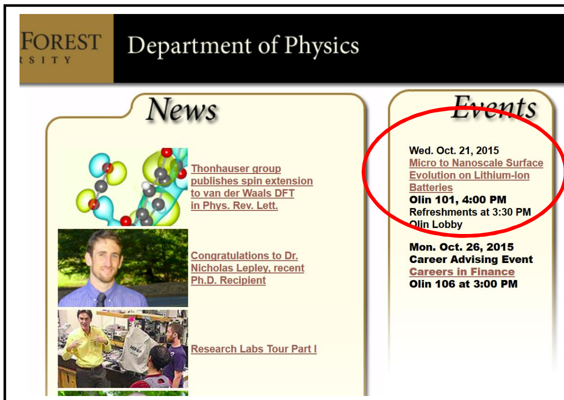
PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 22:

Rotational motion (Chapter 5)

1. Rigid body motion in body fixed frame
2. Conversion between body and inertial reference frames
3. Symmetric top motion

10	Wed, 9/16/2015	Chap. 3 & 6	Lagrangians and constants of motion	#10
11	Fri, 9/18/2015	Chap. 3 & 6	Hamiltonian formalism	#11
12	Mon, 9/21/2015	Chap. 3 & 6	Hamiltonian formalism	#12
13	Wed, 9/23/2015	Chap. 3 & 6	Hamiltonian Jacobi transformations	#13
14	Fri, 9/25/2015	Chap. 4	Small oscillations	#14
15	Mon, 9/28/2015	Chap. 4	Normal modes of motion	#15
16	Wed, 9/30/2015	Chap. 7	Wave motion	#16
17	Fri, 10/02/2015	Chap. 7 & App. A	Contour Integration	#17
18	Mon, 10/05/2015	Chap. 7	Fourier transforms	#18
19	Wed, 10/07/2015	Chap. 7	Laplace transforms	#19
20	Fri, 10/09/2015	Chap. 7	Green's functions	Start exam
	Mon, 10/12/2015		No class	Take home exam
	Wed, 10/14/2015		No class	Exam due before 10/19/2015
	Fri, 10/16/2015		Fall break -- no class	
21	Mon, 10/19/2015	Chap. 5	Motion of Rigid Bodies	#20
22	Wed, 10/21/2015	Chap. 5	Motion of Rigid Bodies	#21
	Wed, 12/02/2015		Student presentations I	
	Fri, 12/04/2015		Student presentations II	
	Mon, 12/07/2015		Begin Take-home final	



WFU Physics Colloquium

TITLE: Micro to Nanoscale Surface Evolution on Lithium-Ion Batteries

SPEAKER: Sung-Jin Cho,

Department of Nano Engineering,
Joint School of Nanoscience and Nanoengineering,
North Carolina A&T State University,
Greensboro, NC

TIME: Wednesday October 21, 2015 at 4:00 PM

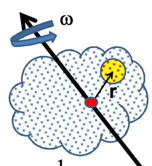
PLACE: Room 101 Olin Physical Laboratory

Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

ABSTRACT

Two ongoing research projects related to energy storage materials and devices such as lithium-ion batteries will be described. First, we report a thoroughly new concept applied to the design and synthesis of a multifunctional, nanoscale interface and multi-phase

Summary of previous results describing rigid bodies rotating about a fixed origin ●



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega}\times\mathbf{r}$$

Kinetic energy: $T = \sum_p \frac{1}{2} m_p v_p^2 = \sum_p \frac{1}{2} m_p \left(\left|\boldsymbol{\omega}\times\mathbf{r}_p\right|\right)^2$

$$= \sum_p \frac{1}{2} m_p \left(\boldsymbol{\omega}\times\mathbf{r}_p\right)\cdot\left(\boldsymbol{\omega}\times\mathbf{r}_p\right)$$

$$= \sum_p \frac{1}{2} m_p \left[\left(\boldsymbol{\omega}\cdot\boldsymbol{\omega}\right)\left(\mathbf{r}_p\cdot\mathbf{r}_p\right) - \left(\mathbf{r}_p\cdot\boldsymbol{\omega}\right)^2\right]$$

$$= \boldsymbol{\omega}\cdot\ddot{\mathbf{I}}\cdot\boldsymbol{\omega} \qquad \ddot{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p\mathbf{r}_p\right)$$

Moment of inertia tensor

Matrix notation:

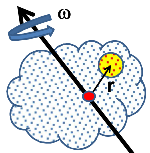
$$\ddot{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \qquad I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj}\right)$$

For general coordinate system: $T = \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j$

For (body fixed) coordinate system that diagonalizes moment of inertia tensor: $\ddot{\mathbf{I}}\cdot\hat{\mathbf{e}}_i = I_i\hat{\mathbf{e}}_i \qquad i = 1,2,3$

$$\boldsymbol{\omega} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3 \qquad \Rightarrow T = \frac{1}{2}\sum_i I_i\tilde{\omega}_i^2$$

Summary of previous results describing rigid bodies rotating about a fixed origin



$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

Angular momentum: $\mathbf{L} = \sum_p m_p \mathbf{r}_p \times \mathbf{v}_p = \sum_p m_p \mathbf{r}_p \times \left(\boldsymbol{\omega} \times \mathbf{r}_p\right)$

$$= \sum_p m_p \left[\boldsymbol{\omega}\left(\mathbf{r}_p \cdot \mathbf{r}_p\right) - \mathbf{r}_p\left(\mathbf{r}_p \cdot \boldsymbol{\omega}\right)\right]$$

$$= \ddot{\mathbf{I}} \cdot \boldsymbol{\omega} \qquad \ddot{\mathbf{I}} \equiv \sum_p m_p \left(\mathbf{1} r_p^2 - \mathbf{r}_p \mathbf{r}_p\right)$$

Descriptions of rotation about a given origin -- continued

For (body fixed) coordinate system that diagonalizes moment of inertia tensor:

$$\ddot{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad \boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

Time derivative: $\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L}$

$$\frac{d\mathbf{L}}{dt} = I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 +$$
$$\tilde{\omega}_2 \tilde{\omega}_3 \left(I_3 - I_2\right) \hat{\mathbf{e}}_1 + \tilde{\omega}_3 \tilde{\omega}_1 \left(I_1 - I_3\right) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 \left(I_2 - I_1\right) \hat{\mathbf{e}}_3$$

Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.

For $\boldsymbol{\tau} = 0$ we can solve the Euler equations:

$$\frac{d\mathbf{L}}{dt} = 0 = I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 +$$
$$\tilde{\omega}_2 \tilde{\omega}_3 \left(I_3 - I_2\right) \hat{\mathbf{e}}_1 + \tilde{\omega}_3 \tilde{\omega}_1 \left(I_1 - I_3\right) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 \left(I_2 - I_1\right) \hat{\mathbf{e}}_3$$

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 \left(I_3 - I_2\right) = 0$$

$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 \left(I_1 - I_3\right) = 0$$

$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 \left(I_2 - I_1\right) = 0$$

Euler equations for rotation in body fixed frame :

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3(I_3 - I_2) = 0$$
$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1(I_1 - I_3) = 0$$
$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2(I_2 - I_1) = 0$$

Solution for symmetric top -- $I_2 = I_1$:

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3(I_3 - I_1) = 0$$
$$I_1\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1(I_1 - I_3) = 0$$
$$I_3\dot{\tilde{\omega}}_3 = 0 \qquad \Rightarrow \tilde{\omega}_3 = \text{(constant)}$$

Define : $\Omega \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_1}$

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2\Omega$$
$$\dot{\tilde{\omega}}_2 = \tilde{\omega}_1\Omega$$

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Solution of Euler equations for a symmetric top -- continued

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2\Omega \qquad \dot{\tilde{\omega}}_2 = \tilde{\omega}_1\Omega$$

where $\Omega \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_1}$

Solution : $\tilde{\omega}_1(t) = A\cos(\Omega t + \varphi)$

$\tilde{\omega}_2(t) = A\sin(\Omega t + \varphi)$

$$T = \frac{1}{2}\sum_i I_i\tilde{\omega}_i^2 = \frac{1}{2}I_1 A^2 + \frac{1}{2}I_3\tilde{\omega}_3^2$$

$$\mathbf{L} = I_1\tilde{\omega}_1\hat{\mathbf{e}}_1 + I_2\tilde{\omega}_2\hat{\mathbf{e}}_2 + I_3\tilde{\omega}_3\hat{\mathbf{e}}_3$$
$$= I_1 A\left(\cos(\Omega t + \varphi)\hat{\mathbf{e}}_1 + \sin(\Omega t + \varphi)\hat{\mathbf{e}}_2\right) + I_3\tilde{\omega}_3\hat{\mathbf{e}}_3$$

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Euler equations for rotation in body fixed frame :

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3(I_3 - I_2) = 0$$
$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1(I_1 - I_3) = 0$$
$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2(I_2 - I_1) = 0$$

Solution for asymmetric top -- $I_3 \neq I_2 \neq I_1$:

Suppose : $\dot{\tilde{\omega}}_3 \approx 0$ Define : $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$

Define : $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

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Euler equations for asymmetric top -- continued

$$I_1\dot{\tilde{\omega}}_1 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right) = 0$$
$$I_2\dot{\tilde{\omega}}_2 + \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right) = 0$$
$$I_3\dot{\tilde{\omega}}_3 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right) = 0$$

If $\dot{\tilde{\omega}}_3 \approx 0$, Define: $\Omega_1 \equiv \tilde{\omega}_3 \frac{I_3 - I_2}{I_1}$ $\Omega_2 \equiv \tilde{\omega}_3 \frac{I_3 - I_1}{I_2}$

$$\dot{\tilde{\omega}}_1 = -\Omega_1\tilde{\omega}_2 \qquad \dot{\tilde{\omega}}_2 = \Omega_2\tilde{\omega}_1$$

If Ω_1 and Ω_2 are both positive or both negative :

$$\tilde{\omega}_1(t) \approx A\cos\left(\sqrt{\Omega_1\Omega_2}\,t + \varphi\right)$$
$$\tilde{\omega}_2(t) \approx A\sqrt{\frac{\Omega_2}{\Omega_1}}\sin\left(\sqrt{\Omega_1\Omega_2}\,t + \varphi\right)$$

$\Rightarrow$ If Ω_1 and Ω_2 have opposite signs, solution is unstable.

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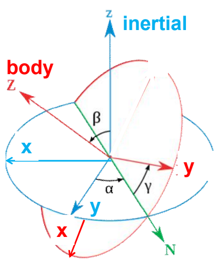
Transformation between body-fixed and inertial coordinate systems – Euler angles



http://en.wikipedia.org/wiki/Euler_angles

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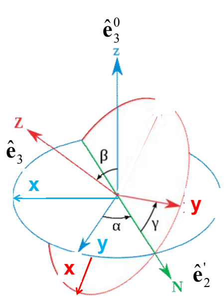

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2' + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

Need to express all components in body-fixed frame:

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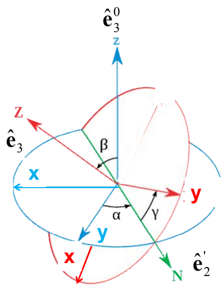
$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



$$\hat{\mathbf{e}}_2^{'} = \sin\gamma\,\hat{\mathbf{e}}_1 + \cos\gamma\,\hat{\mathbf{e}}_2$$

Matrix representation :

$$\hat{\mathbf{e}}_2^{'} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

$$\hat{\mathbf{e}}_3^0 = -\sin\beta\;\hat{\mathbf{e}}'_1 + \cos\beta\;\hat{\mathbf{e}}'_3 = -\cos\gamma\sin\beta\;\hat{\mathbf{e}}_1 + \sin\gamma\sin\beta\;\hat{\mathbf{e}}_2 + \cos\beta\;\hat{\mathbf{e}}_3$$

Matrix representation:

$$\hat{\mathbf{e}}_3^0 = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2^{'} + \dot{\gamma}\,\hat{\mathbf{e}}_3$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix} + \dot{\beta}\begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix} + \dot{\gamma}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

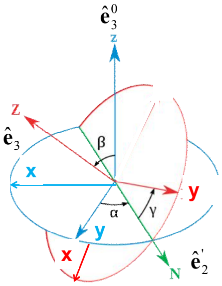
$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\begin{pmatrix} -\sin\beta\cos\gamma \\ \sin\beta\sin\gamma \\ \cos\beta \end{pmatrix} + \dot{\beta}\begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix} + \dot{\gamma}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{\omega}_1 = \dot{\alpha}(-\sin\beta\cos\gamma) + \dot{\beta}\sin\gamma$$

$$\tilde{\omega}_2 = \dot{\alpha}(\sin\beta\sin\gamma) + \dot{\beta}\cos\gamma$$

$$\tilde{\omega}_3 = \dot{\alpha}\cos\beta + \dot{\gamma}$$

$$\tilde{\boldsymbol{\omega}} = \dot{\alpha}\,\hat{\mathbf{e}}_3^0 + \dot{\beta}\,\hat{\mathbf{e}}_2' + \dot{\gamma}\,\hat{\mathbf{e}}_3$$



$$\begin{aligned}\tilde{\boldsymbol{\omega}} = &\left[\dot{\alpha}(-\sin\beta\cos\gamma)+\dot{\beta}\sin\gamma\right]\hat{\mathbf{e}}_1\\ &+\left[\dot{\alpha}(\sin\beta\sin\gamma)+\dot{\beta}\cos\gamma\right]\hat{\mathbf{e}}_2\\ &+\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]\hat{\mathbf{e}}_3\end{aligned}$$

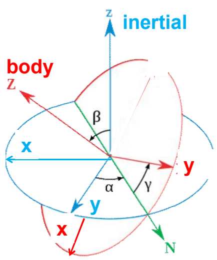
Rotational kinetic energy

$$\begin{aligned}T(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) &= \frac{1}{2}I_1\tilde{\omega}_1^2+\frac{1}{2}I_2\tilde{\omega}_2^2+\frac{1}{2}I_3\tilde{\omega}_3^2\\ &=\frac{1}{2}I_1\left[\dot{\alpha}(-\sin\beta\cos\gamma)+\dot{\beta}\sin\gamma\right]^2\\ &+\frac{1}{2}I_2\left[\dot{\alpha}(\sin\beta\sin\gamma)+\dot{\beta}\cos\gamma\right]^2\\ &+\frac{1}{2}I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]^2\end{aligned}$$

If $\quad I_1 = I_2$:

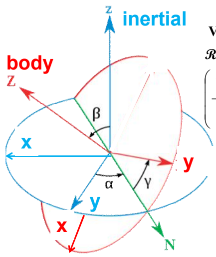
$$T(\alpha,\beta,\gamma,\dot{\alpha},\dot{\beta},\dot{\gamma}) = \frac{1}{2}I_1\left(\dot{\alpha}^2\sin^2\beta+\dot{\beta}^2\right)+\frac{1}{2}I_3\left[\dot{\alpha}\cos\beta+\dot{\gamma}\right]^2$$

Transformation between body-fixed and inertial coordinate systems – Euler angles



http://en.wikipedia.org/wiki/Euler_angles

General transformation between rotated coordinates – Euler angles

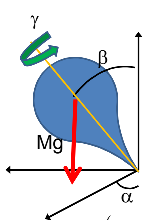


$$\mathbf{V}' = \mathcal{R}\mathbf{V} = \mathcal{R}_\alpha \mathcal{R}_\beta \mathcal{R}_\gamma \mathbf{V}$$

$$\mathcal{R} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

http://en.wikipedia.org/wiki/Euler_angles

Motion of a symmetric top under the influence of the torque of gravity:



$$L(\alpha,\beta,\gamma,\dot\alpha,\dot\beta,\dot\gamma) = \frac{1}{2}I_1\left(\dot\alpha^2\sin^2\beta + \dot\beta^2\right) + \frac{1}{2}I_3\left[\dot\alpha\cos\beta + \dot\gamma\right]^2 - Mgl\cos\beta$$

$$L(\alpha,\beta,\gamma,\dot\alpha,\dot\beta,\dot\gamma) = \frac{1}{2}I_1\left(\dot\alpha^2\sin^2\beta + \dot\beta^2\right) + \frac{1}{2}I_3\left[\dot\alpha\cos\beta + \dot\gamma\right]^2 - Mgl\cos\beta$$

Constants of the motion :

$$p_\alpha = \frac{\partial L}{\partial \dot\alpha} = I_1\dot\alpha\sin^2\beta + I_3\left[\dot\alpha\cos\beta + \dot\gamma\right]\cos\beta$$

$$p_\gamma = \frac{\partial L}{\partial \dot\gamma} = I_3\left[\dot\alpha\cos\beta + \dot\gamma\right]$$

$$E = \frac{1}{2}I_1\dot\beta^2 + \frac{p_\gamma^2}{2I_3} + V_{eff}(\beta)$$

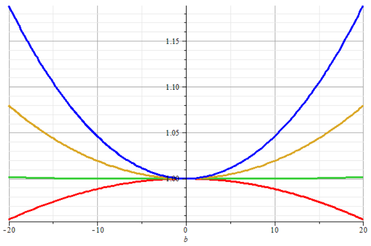
$$L(\beta,\dot\beta) = \frac{1}{2}I_1\dot\beta^2 + \frac{(p_\alpha - p_\gamma\cos\beta)^2}{2I_1\sin^2\beta} + \frac{p_\gamma^2}{2I_3} - Mgl\cos\beta$$

$$V_{eff}(\beta) = \frac{(p_\alpha - p_\gamma\cos\beta)^2}{2I_1\sin^2\beta} + Mgl\cos\beta$$

$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

Stable/unstable solutions near β=0



Suppose $p_\alpha = p_\gamma$ and $\beta \approx 0$

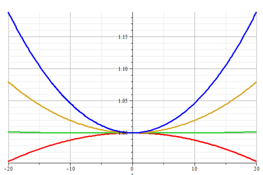
$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$

$$E' \approx \frac{1}{2} I_1 \dot{\beta}^2 + \frac{{p_\gamma}^2}{2I_1} \frac{\left(1 - 1 + \frac{1}{2}\beta^2\right)^2}{\beta^2} + Mgl\left(1 - \frac{1}{2}\beta^2\right)$$

$$\approx \frac{1}{2} I_1 \dot{\beta}^2 + \left(\frac{{p_\gamma}^2}{8I_1} - \frac{Mgl}{2}\right)\beta^2 + Mgl$$

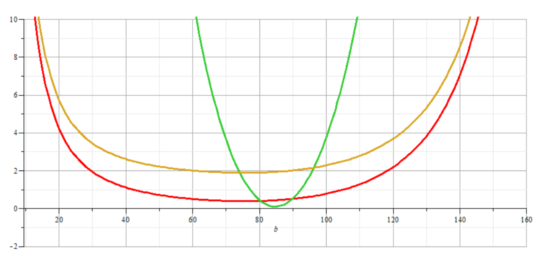
$\Rightarrow$ Stable solution if

$$p_\gamma \geq \sqrt{4MglI_1}$$



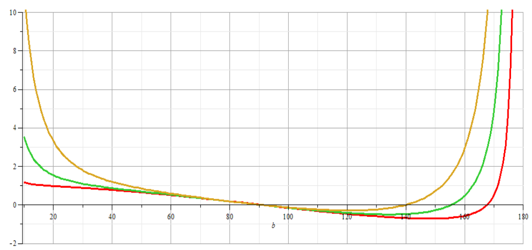
More general case:

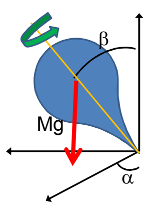
$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl\cos\beta$$



More general case:

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2I_1 \sin^2\beta} + Mgl \cos\beta$$





Constants of the motion :

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3[\dot{\alpha}\cos\beta + \dot{\gamma}]$$

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1 \dot{\alpha} \sin^2\beta + I_3[\dot{\alpha}\cos\beta + \dot{\gamma}]\cos\beta$$

$$= I_1 \dot{\alpha} \sin^2\beta + p_\gamma \cos\beta$$