

**PHY 711 Classical Mechanics and  
Mathematical Methods**  
**10-10:50 AM MWF Olin 103**

**Plan for Lecture 24:**

**Introduction to hydrodynamics  
(Chap. 9 in F & W)**

- 1. Motivation for topic**
- 2. Newton's laws for fluids**
- 3. Conservation relations**

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|    |                 |                  |                                                         |
|----|-----------------|------------------|---------------------------------------------------------|
| 10 | Wed, 9/16/2015  | Chap. 3 & 6      | Lagrangians and constants of motion <a href="#">#10</a> |
| 11 | Fri, 9/18/2015  | Chap. 3 & 6      | Hamiltonian formalism <a href="#">#11</a>               |
| 12 | Mon, 9/21/2015  | Chap. 3 & 6      | Hamiltonian formalism <a href="#">#12</a>               |
| 13 | Wed, 9/23/2015  | Chap. 3 & 6      | Hamiltonian Jacobi transformations <a href="#">#13</a>  |
| 14 | Fri, 9/25/2015  | Chap. 4          | Small oscillations <a href="#">#14</a>                  |
| 15 | Mon, 9/28/2015  | Chap. 4          | Normal modes of motion <a href="#">#15</a>              |
| 16 | Wed, 9/30/2015  | Chap. 7          | Wave motion <a href="#">#16</a>                         |
| 17 | Fri, 10/02/2015 | Chap. 7 & App. A | Contour integration <a href="#">#17</a>                 |
| 18 | Mon, 10/05/2015 | Chap. 7          | Fourier transforms <a href="#">#18</a>                  |
| 19 | Wed, 10/07/2015 | Chap. 7          | Laplace transforms <a href="#">#19</a>                  |
| 20 | Fri, 10/09/2015 | Chap. 7          | Green's functions Start exam                            |
|    | Mon, 10/12/2015 |                  | No class Take home exam                                 |
|    | Wed, 10/14/2015 |                  | No class Exam due before 10/19/2015                     |
|    | Fri, 10/16/2015 |                  | Fall break -- no class                                  |
| 21 | Mon, 10/19/2015 | Chap. 5          | Motion of Rigid Bodies <a href="#">#20</a>              |
| 22 | Wed, 10/21/2015 | Chap. 5          | Motion of Rigid Bodies <a href="#">#21</a>              |
| 23 | Fri, 10/23/2015 | Chap. 8          | Motion of Elastic membranes <a href="#">#22</a>         |
| 24 | Mon, 10/26/2015 | Chap. 9          | Hydrodynamics <a href="#">#23</a>                       |
|    | Wed, 12/02/2015 |                  | Student presentations I                                 |
|    | Fri, 12/04/2015 |                  | Student presentations II                                |
|    | Mon, 12/07/2015 |                  | Begin Take-home final                                   |

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**Motivation**

1. Natural progression from strings, membranes, fluids; description of 1, 2, and 3 dimensional continua
2. Interesting and technologically important phenomena associated with fluids

**Plan**

1. Newton's laws for fluids
2. Continuity equation
3. Stress tensor
4. Energy relations
5. Bernoulli's theorem
6. Various examples
7. Sound waves

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## Newton's equations for fluids

Use Lagrange formulation; following "particles" of fluid

Variables: Density  $\rho(x,y,z,t)$ Pressure  $p(x,y,z,t)$ Velocity  $\mathbf{v}(x,y,z,t)$ 

$$m\mathbf{a} = \mathbf{F}$$

$$m \rightarrow \rho dV$$

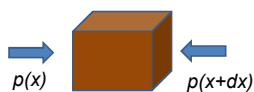
$$\mathbf{a} \rightarrow \frac{d\mathbf{v}}{dt}$$

$$\mathbf{F} \rightarrow \mathbf{F}_{\text{applied}} + \mathbf{F}_{\text{pressure}}$$

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## Newton's equations for fluids -- continued

$$m\mathbf{a} = \mathbf{F}_{\text{applied}} + \mathbf{F}_{\text{pressure}}$$

$$\rho dV \frac{d\mathbf{v}}{dt} = \mathbf{f}_{\text{applied}} \rho dV - (\nabla p) dV$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

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Detailed analysis of acceleration term:

$$\mathbf{v} = \mathbf{v}(x, y, z, t)$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x} \frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y} \frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z} \frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x} v_x + \frac{\partial \mathbf{v}}{\partial y} v_y + \frac{\partial \mathbf{v}}{\partial z} v_z + \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{d\mathbf{v}}{dt} = (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}$$

Note that :

$$\frac{\partial \mathbf{v}}{\partial x} v_x + \frac{\partial \mathbf{v}}{\partial y} v_y + \frac{\partial \mathbf{v}}{\partial z} v_z = \nabla \left( \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) - \mathbf{v} \times (\nabla \times \mathbf{v})$$

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Newton's equations for fluids -- continued

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

$$\rho \left( \nabla \left( \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) + \frac{\partial \mathbf{v}}{\partial t} \right) = \rho \mathbf{f}_{\text{applied}} - \nabla p$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left( \frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

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Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left( \frac{1}{2} v^2 \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1.  $(\nabla \times \mathbf{v}) = 0$  "irrotational flow"

$$\Rightarrow \mathbf{v} = -\nabla \Phi$$

2.  $\mathbf{f}_{\text{applied}} = -\nabla U$  conservative applied force

3.  $\rho = (\text{constant})$  incompressible fluid

$$\frac{\partial(-\nabla \Phi)}{\partial t} + \nabla \left( \frac{1}{2} v^2 \right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla \left( \frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} \right) = 0$$

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## Bernoulli's integral of Euler's equation

$$\nabla \left( \frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} \right) = 0$$

Integrating over space:

$$\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} = C(t)$$

where  $\mathbf{v} = -\nabla \Phi(\mathbf{r}, t) = -\nabla (\Phi(\mathbf{r}, t) + C'(t))$ 

$$\Rightarrow \frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} = 0 \quad \text{Bernoulli's theorem}$$

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## Examples of Bernoulli's theorem

$$\frac{p}{\rho} + U + \frac{1}{2} v^2 - \frac{\partial \Phi}{\partial t} = 0$$

Modified form; assuming  $\frac{\partial \Phi}{\partial t} = 0$ 

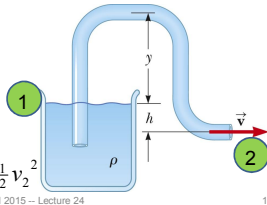
$$\frac{p}{\rho} + U + \frac{1}{2} v^2 = \text{constant}$$

$$p_1 = p_2 = p_{atm}$$

$$U_1 - U_2 = gh$$

$$v_1 \approx 0$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2} v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2} v_2^2$$



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## Examples of Bernoulli's theorem -- continued

$$p_1 = p_2 = p_{atm}$$

$$U_1 - U_2 = gh$$

$$v_1 \approx 0$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2} v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2} v_2^2$$

$$v_2 \approx \sqrt{2gh}$$

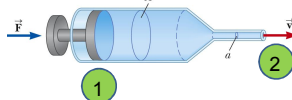
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## Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \frac{1}{2} v^2 = \text{constant}$$



$$p_1 = \frac{F}{A} + p_{atm} \quad p_2 = p_{atm}$$

$$U_1 = U_2$$

$$v_1 A = v_2 a \quad \text{continuity equation}$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2} v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2} v_2^2$$

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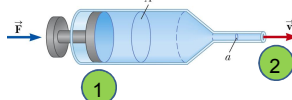
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## Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \frac{1}{2} v^2 = \text{constant}$$



$$\frac{2F}{A} = v_2^2 \left( 1 - \left( \frac{a}{A} \right)^2 \right)$$

$$v_2 = \sqrt{\frac{2F/A}{1 - \left( \frac{a}{A} \right)^2}}$$

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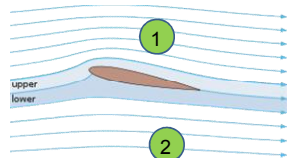
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## Examples of Bernoulli's theorem -- continued

## Approximate explanation of airplane lift

Cross section view of airplane wing

[http://en.wikipedia.org/wiki/Lift\\_%28force%29](http://en.wikipedia.org/wiki/Lift_%28force%29)


$$U_1 \approx U_2$$

$$\frac{p_1}{\rho} + U_1 + \frac{1}{2} v_1^2 = \frac{p_2}{\rho} + U_2 + \frac{1}{2} v_2^2$$

$$p_2 - p_1 = \frac{1}{2} (v_1^2 - v_2^2)$$

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Continuity equation connecting fluid density and velocity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

Consider:  $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\nabla \rho) \cdot \mathbf{v}$

$$\Rightarrow \frac{d\rho}{dt} + \rho(\nabla \cdot \mathbf{v}) = 0 \quad \text{alternative form of continuity equation}$$

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Some details on the velocity potential

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

For incompressible fluid:  $\rho = (\text{constant})$

$$\Rightarrow \nabla \cdot \mathbf{v} = 0$$

Irrotational flow:  $\nabla \times \mathbf{v} = 0 \quad \Rightarrow \mathbf{v} = -\nabla \Phi$

$$\Rightarrow \nabla^2 \Phi = 0$$

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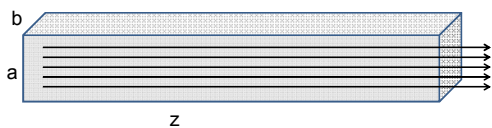
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Example – uniform flow



$$\nabla^2 \Phi = 0$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Possible solution:

$$\Phi = -v_0 z$$

$$\mathbf{v} = -\nabla \Phi = v_0 \hat{\mathbf{z}}$$

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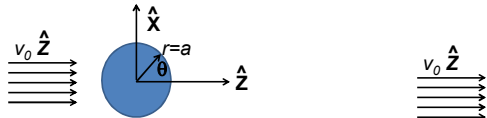
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Example – flow around a long cylinder (oriented in the  $Y$  direction)



$$\nabla^2 \Phi = 0$$

$$\left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 0$$

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Laplace equation in cylindrical coordinates

( $r, \theta$ , defined in  $x$ - $z$  plane;  $y$  representing cylinder axis)

$$\nabla^2 \Phi = 0 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

In our case, there is no motion in the  $y$  dimension

$$\Rightarrow \Phi(r, \theta, y) = \Phi(r, \theta)$$

From boundary condition :  $v_z(r \rightarrow \infty) = v_0$

$$\frac{\partial \Phi}{\partial z}(r \rightarrow \infty) = -v_0 \quad \Rightarrow \quad \Phi(r \rightarrow \infty, \theta) = -v_0 r \cos \theta$$

$$\text{Note that : } \frac{\partial^2 \cos \theta}{\partial \theta^2} = -\cos \theta$$

$$\text{Guess form : } \Phi(r, \theta) = f(r) \cos \theta$$

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Necessary equation for radial function

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial f}{\partial r} - \frac{1}{r^2} f = 0$$

$$f(r) = Ar + \frac{B}{r} \quad \text{where } A, B \text{ are constants}$$

Boundary condition on cylinder surface :

$$\left. \frac{\partial \Phi}{\partial r} \right|_{r=a} = 0$$

$$\frac{df}{dr}(r=a) = 0 = A - \frac{B}{a^2}$$

$$\Rightarrow B = Aa^2$$

Boundary condition at  $\infty$  :  $\Rightarrow A = -v_0$

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$$\Phi(r, \theta) = -v_0 \left( r + \frac{a^2}{r} \right) \cos \theta$$

$$v_r = -\frac{\partial \Phi}{\partial r} = v_0 \left( 1 - \frac{a^2}{r^2} \right) \cos \theta$$

$$v_\theta = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta} = v_0 \left( 1 + \frac{a^2}{r^2} \right) \sin \theta$$

For 3-dimensional system, consider a spherical obstruction

Laplacian in spherical polar coordinates:

$$\nabla^2 \Phi = 0 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \varphi^2}$$

to be continued ...

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