

**PHY 711 Classical Mechanics and
Mathematical Methods**
10-10:50 AM MWF Olin 103

Plan for Lecture 26: Chap. 9 of F&W

**Wave equation for sound in linear
approximation**

- 1. Wave equations for sound**
- 2. Speed of sound**
- 3. Standing wave solutions**

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12 Mon, 9/21/2015	Chap. 3 & 6	Hamiltonian formalism	#12
13 Wed, 9/23/2015	Chap. 3 & 6	Hamiltonian Jacobi transformations	#13
14 Fri, 9/25/2015	Chap. 4	Small oscillations	#14
15 Mon, 9/28/2015	Chap. 4	Normal modes of motion	#15
16 Wed, 9/30/2015	Chap. 7	Wave motion	#16
17 Fri, 10/02/2015	Chap. 7 & App. A	Contour Integration	#17
18 Mon, 10/05/2015	Chap. 7	Fourier transforms	#18
19 Wed, 10/07/2015	Chap. 7	Laplace transforms	#19
20 Fri, 10/09/2015	Chap. 7	Green's functions	Start exam
Mon, 10/12/2015		No class	Take home exam
Wed, 10/14/2015		No class	Exam due before 10/19/2015
Fri, 10/16/2015		Fall break -- no class	
21 Mon, 10/19/2015	Chap. 5	Motion of Rigid Bodies	#20
22 Wed, 10/21/2015	Chap. 5	Motion of Rigid Bodies	#21
23 Fri, 10/23/2015	Chap. 8	Motion of Elastic membranes	#22
24 Mon, 10/26/2015	Chap. 9	Hydrodynamics	#23
25 Wed, 10/28/2015	Chap. 9	Hydrodynamics	#24
26 Fri, 10/30/2015	Chap. 9	Sound waves	#25
Wed, 12/02/2015		Student presentations I	
Fri, 12/04/2015		Student presentations II	
Mon, 12/07/2015		Begin Take-home final	

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Application of fluid equations to the case of air in
equilibrium plus small perturbation

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

Near equilibrium :

$$\rho = \rho_0 + \delta \rho$$

$$p = p_0 + \delta p$$

$$\mathbf{v} = 0 + \delta \mathbf{v}$$

$$\mathbf{f}_{\text{applied}} = 0$$

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Equations to lowest order in perturbation :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{f}_{\text{applied}} - \frac{\nabla p}{\rho} \quad \Rightarrow \quad \frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \Rightarrow \quad \frac{\partial \delta \rho}{\partial t} + \rho_0 \nabla \cdot \delta \mathbf{v} = 0$$

In terms of the velocity potential :

$$\delta \mathbf{v} = -\nabla \Phi$$

$$\frac{\partial \delta \mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0} \quad \Rightarrow \quad \nabla \left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0} \right) = 0$$

$$\frac{\partial \delta \rho}{\partial t} + \rho_0 \nabla \cdot \delta \mathbf{v} = 0 \quad \Rightarrow \quad \frac{\partial \delta \rho}{\partial t} - \rho_0 \nabla^2 \Phi = 0$$

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Expressing pressure in terms of the density :

$$p = p(s, \rho) = p_0 + \delta p \quad \text{where } s \text{ denotes the (constant) entropy}$$

$$p_0 = p(s, \rho_0)$$

$$\delta p = \left(\frac{\partial p}{\partial \rho} \right)_s \delta \rho \equiv c^2 \delta \rho$$

$$\nabla \left(-\frac{\partial \Phi}{\partial t} + \frac{\delta p}{\rho_0} \right) = 0 \quad \Rightarrow \quad -\frac{\partial \Phi}{\partial t} + c^2 \frac{\delta \rho}{\rho_0} = (\text{constant})$$

$$\Rightarrow -\frac{\partial^2 \Phi}{\partial t^2} + \frac{c^2}{\rho_0} \frac{\partial \delta \rho}{\partial t} = 0$$

$$\frac{\partial \delta \rho}{\partial t} - \rho_0 \nabla^2 \Phi = 0 \quad \Rightarrow \quad \frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

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Wave equation for air :

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

$$\text{Here, } c^2 = \left(\frac{\partial p}{\partial \rho} \right)_s$$

$$\mathbf{v} = -\nabla \Phi$$

Note that, we also have :

$$\frac{\partial^2 \delta \rho}{\partial t^2} - c^2 \nabla^2 \delta \rho = 0$$

$$\frac{\partial^2 \delta p}{\partial t^2} - c^2 \nabla^2 \delta p = 0$$

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Analysis of wave velocity in an ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho} \right)_s \quad \leftarrow \text{adiabatic conditions}$$

Equation of state for ideal gas :

$$pV = NkT \quad N = \frac{M}{M_0}$$

$$p = \frac{M}{V} \frac{k}{M_0} T = \rho \frac{k}{M_0} T$$

$$k = 1.38 \times 10^{-23} \text{ J / K}$$

M_0 = average mass of each molecule

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Internal energy for ideal gas :

$$E = \frac{f}{2} NkT = M\varepsilon \quad \varepsilon = \frac{f}{2} \frac{k}{M_0} T = \frac{f}{2} \frac{p}{\rho}$$

In terms of specific heat ratio : $\gamma \equiv \frac{C_p}{C_v}$

$$dE = dQ - dW$$

$$C_v = \left(\frac{dQ}{dT} \right)_v = \left(\frac{\partial E}{\partial T} \right)_v = \frac{f}{2} \frac{Mk}{M_0}$$

$$C_p = \left(\frac{dQ}{dT} \right)_p = \left(\frac{\partial E}{\partial T} \right)_p + p \left(\frac{\partial V}{\partial T} \right)_p = \frac{f}{2} \frac{Mk}{M_0} + \frac{Mk}{M_0}$$

$$\gamma = \frac{C_p}{C_v} = \frac{\frac{f}{2} + 1}{\frac{f}{2}} \quad \Rightarrow \quad \frac{f}{2} = \frac{1}{\gamma - 1}$$

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Internal energy for ideal gas :

$$E = \frac{1}{\gamma - 1} NkT = M\varepsilon \quad \varepsilon = \frac{1}{\gamma - 1} \frac{k}{M_0} T = \frac{1}{\gamma - 1} \frac{p}{\rho}$$

Internal energy for ideal gas under isentropic conditions :

$$d\varepsilon = -\frac{p}{M} dV = \frac{p}{\rho^2} d\rho$$

$$\left(\frac{\partial \varepsilon}{\partial \rho} \right)_s = \frac{p}{\rho^2} = \frac{\partial}{\partial \rho} \left(\frac{1}{\gamma - 1} \frac{p}{\rho} \right)_s = \left(\frac{\partial p}{\partial \rho} \right)_s \frac{1}{(\gamma - 1)\rho} - \frac{p}{(\gamma - 1)\rho^2}$$

$$\Rightarrow \left(\frac{\partial p}{\partial \rho} \right)_s = \frac{p\gamma}{\rho}$$

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Alternative derivation:

Isentropic or adiabatic equation of state:

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \quad \Rightarrow \quad \frac{p}{p_0} = \left(\frac{\rho}{\rho_0} \right)^\gamma$$

$$\left(\frac{\partial p}{\partial \rho} \right)_s = \frac{p\gamma}{\rho}$$

Linearized speed of sound

$$c_0^2 = \left(\frac{\partial p}{\partial \rho} \right)_{s, p_0, \rho_0} = \frac{p_0 \gamma}{\rho_0}$$

$$c_0^2 \approx \frac{1.5 \cdot 1.0 \times 10^5 \text{ Pa}}{1.3 \text{ kg/m}^3} \quad c_0 \approx 340 \text{ m/s}$$

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Density dependence of speed of sound for ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho} \right)_s = \frac{p\gamma}{\rho}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0} \right)^\gamma$$

$$c^2 = \frac{p_0 \gamma}{\rho_0} \frac{p / p_0}{\rho / \rho_0} = c_0^2 \left(\frac{\rho}{\rho_0} \right)^{\gamma-1}$$

where $c_0^2 = \frac{p_0 \gamma}{\rho_0}$

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Wave equation for air:

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho} \right)_s$

$$\mathbf{v} = -\nabla \Phi$$

Boundary values:

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta \mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla \Phi$$

Free surface:

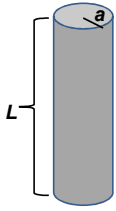
$$\delta p = 0 \quad \Rightarrow \quad \rho_0 \frac{\partial \Phi}{\partial t} = 0$$

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Time harmonic standing waves in a pipe



$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Boundary values :

At fixed surface : $\hat{\mathbf{n}} \cdot \nabla \Phi = 0$

At free surface : $\frac{\partial \Phi}{\partial t} = 0$

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$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0 \quad \text{Define : } k \equiv \frac{\omega}{c}$$

In cylindrical coordinates :

$$\Phi(r, \varphi, z, t) = R(r)F(\varphi)Z(z)e^{-i\omega t} \equiv R(r)F(\varphi)Z(z)e^{-ikct}$$

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \Phi(r, \varphi, z, t) = 0$$

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$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \Phi(r, \varphi, z, t) = 0$$

$$\Phi(r, \varphi, z, t) = R(r)F(\varphi)Z(z)e^{-i\omega t}$$

$$F(\varphi) = e^{im\varphi}; F(\varphi) = F(\varphi + 2\pi N) \Rightarrow m = \text{integer}$$

$$Z(z) = e^{i\alpha z}; \alpha = \text{real plus other restrictions}$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - \alpha^2 + k^2 \right) R(r) = 0$$

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$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} - \alpha^2 + k^2 \right) R(r) = 0$$

For $k^2 \geq \alpha^2$ define $\kappa^2 \equiv k^2 - \alpha^2$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + \kappa^2 \right) R(r) = 0$$

Cylinder surface boundary conditions : $\left. \frac{dR}{dr} \right|_{r=a} = 0$

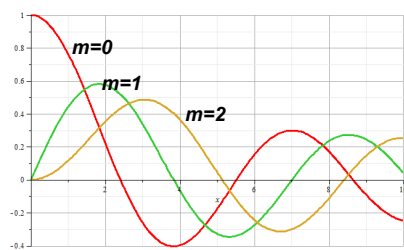
$$\Rightarrow R(r) = J_m(\kappa r) \quad \text{where for } \frac{dJ_m(x')}{dx} = 0, \quad \kappa_{mn} = \frac{x'_{mn}}{a}$$

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Bessel functions : $J_m(x)$

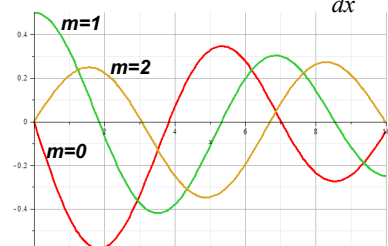


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Bessel function derivatives : $\frac{dJ_m(x)}{dx}$



Zeros of derivatives: m=0: 0.00000, 3.83171, 7.01559
 m=1: 1.84118, 5.33144, 8.53632
 m=2: 3.05424, 6.70613, 9.96947

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Boundary condition for $z=0, z=L$:

For open - open pipe :

$$Z(0) = Z(L) = 0 \Rightarrow Z(z) = \sin\left(\frac{p\pi z}{L}\right)$$

$$\Rightarrow \alpha_p = \frac{p\pi}{L}, \quad p = 1, 2, 3, \dots$$

Resonant frequencies :

$$\frac{\omega^2}{c^2} = k^2 = \kappa_{mn}^2 + \alpha_p^2$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2$$

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Example

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2 = \left(\frac{\pi p}{L}\right)^2 \left(1 + \left(\frac{L}{a}\right)^2 \left(\frac{x'_{mn}}{\pi p}\right)^2\right)$$

$$\pi p = 3.14, 6.28, 9.42, \dots$$

$$x'_{mn} = 0.00, 1.84, 3.05$$

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Alternate boundary condition for $z=0, z=L$:

For open - closed pipe :

$$\frac{dZ(0)}{dz} = Z(L) = 0 \Rightarrow Z(z) = \cos\left(\frac{(2p+1)\pi z}{2L}\right)$$

$$\Rightarrow \alpha_p = \frac{(2p+1)\pi}{2L}, \quad p = 0, 1, 2, 3, \dots$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi(2p+1)}{2L}\right)^2$$

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Other solutions to wave equation:

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0$$

Plane wave solution :

$$\Phi(\mathbf{r}, t) = A e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c} \right)^2$$

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