PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 27: Chap. 9 of F&W

Wave equation for sound in the linear approximation

1. **Sound generation**
2. **Sound scattering**

13	Wed, 9/23/2015	Chap. 3 & 6	Hamiltonian Jacobi transformations	#13
14	Fri, 9/25/2015	Chap. 4	Small oscillations	#14
15	Mon, 9/28/2015	Chap. 4	Normal modes of motion	#15
16	Wed, 9/30/2015	Chap. 7	Wave motion	#16
17	Fri, 10/02/2015	Chap. 7 & App. A	Contour Integration	#17
18	Mon, 10/05/2015	Chap. 7	Fourier transforms	#18
19	Wed, 10/07/2015	Chap. 7	Laplace transforms	#19
20	Fri, 10/09/2015	Chap. 7	Green's functions	Start exam
	Mon, 10/12/2015		No class	Take home exam
	Wed, 10/14/2015		No class	Exam due before 10/19/2015
	Fri, 10/16/2015		Fall break -- no class	
21	Mon, 10/19/2015	Chap. 5	Motion of Rigid Bodies	#20
22	Wed, 10/21/2015	Chap. 5	Motion of Rigid Bodies	#21
23	Fri, 10/23/2015	Chap. 8	Motion of Elastic membranes	#22
24	Mon, 10/26/2015	Chap. 9	Hydrodynamics	#23
25	Wed, 10/28/2015	Chap. 9	Hydrodynamics	#24
26	Fri, 10/30/2015	Chap. 9	Sound waves	#25
27	Mon, 11/02/2015	Chap. 9	Sound waves	#26
	Wed, 12/02/2015		Student presentations I	
	Fri, 12/04/2015		Student presentations II	
	Mon, 12/07/2015		Begin Take-home final	

Solutions to wave equation:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

Plane wave solution :

$$\Phi(\mathbf{r},t) = Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Solution in terms of Green's function :

$$\Phi(\mathbf{r},t) = \int d^3r' \int dt'\, G(\mathbf{r}-\mathbf{r}', t-t')\, f(\mathbf{r}', t')$$

where

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

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Wave equation with source -- continued:

We can show that :

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{\delta\left(t' - \left(t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

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Derivation of Green's function for wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Recall that

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')}\, d\omega$$

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Derivation of Green's function for wave equation -- continued

$$\text{Define: } \quad \tilde{G}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} G(\mathbf{r},t)e^{i\omega t}dt$$

$$G(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(\mathbf{r},\omega)e^{-i\omega t}d\omega$$

$\tilde{G}(\mathbf{r},\omega)$ must satisfy:

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}') \quad \text{where } k^2 = \frac{\omega^2}{c^2}$$

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Derivation of Green's function for wave equation -- continued

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

Solution assuming isotropy in $\mathbf{r}-\mathbf{r}'$:

$$\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Check -- Define $R \equiv |\mathbf{r}-\mathbf{r}'|$ and for $R > 0$:

$$\left(\nabla^2 + k^2\right)\tilde{G}(R,\omega) = \frac{1}{R}\frac{d^2}{dR^2}\left(R\tilde{G}(R,\omega)\right) + k^2\tilde{G}(R,\omega) = 0$$

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Derivation of Green's function for wave equation -- continued

For $R > 0$:

$$\left(\nabla^2 + k^2\right)\tilde{G}(R,\omega) = \frac{1}{R}\frac{d^2}{dR^2}\left(R\tilde{G}(R,\omega)\right) + k^2\tilde{G}(R,\omega) = 0$$

$$\frac{d^2}{dR^2}\left(R\tilde{G}(R,\omega)\right) + k^2\left(R\tilde{G}(R,\omega)\right) = 0$$

$$\left(R\tilde{G}(R,\omega)\right) = A\,e^{ikR} + Be^{-ikR}$$

$$\Rightarrow \tilde{G}(R,\omega) = A\frac{e^{ikR}}{R} + B\frac{e^{-ikR}}{R}$$

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Derivation of Green's function for wave equation – continued
need to find *A* and *B*.

$$\text{Note that: } \nabla^2 \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = -\delta(\mathbf{r}-\mathbf{r}')$$

$$\Rightarrow A = B = \frac{1}{4\pi}$$

$$\tilde{G}(R,\omega) = \frac{e^{\pm ikR}}{4\pi R}$$

Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(\mathbf{r}-\mathbf{r}',\omega)e^{-i\omega(t-t')}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{-i\omega(t-t')}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{-i\omega(t-t')}d\omega$$

Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{-i\omega(t-t')}d\omega$$

$$\text{Noting that } \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega u}d\omega = \delta(u)$$

$$\Rightarrow G(\mathbf{r}-\mathbf{r}',t-t') = \frac{\delta\left(t-\left(t'\mp\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

For time harmonic forcing term we can use the corresponding Green's function:

$$\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)=\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

In fact, this Green's function is appropriate for boundary conditions at infinity. For surface boundary conditions where we know the boundary values or their gradients, the Green's function must be modified.

Green's theorem

Consider two functions $h(\mathbf{r})$ and $g(\mathbf{r})$

Note that : $\int_V \left(h\nabla^2 g - g\nabla^2 h\right) d^3r = \oint_S \left(h\nabla g - g\nabla h\right)\cdot\hat{\mathbf{n}}\, d^2r$

$$\nabla^2\widetilde{\Phi}+k^2\widetilde{\Phi}=-\widetilde{f}(\mathbf{r},\omega)$$

$$\left(\nabla^2+k^2\right)\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega)=-\delta(\mathbf{r}-\mathbf{r}')$$

$h \leftrightarrow \widetilde{\Phi}$; $\quad g \leftrightarrow \widetilde{G}$

$$\int_V \left(\widetilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}')-\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)f(\mathbf{r},\omega)\right)d^3r =$$
$$\oint_S \left(\widetilde{\Phi}(\mathbf{r},\omega)\nabla\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)-\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\widetilde{\Phi}(\mathbf{r},\omega)\right)\cdot\hat{\mathbf{n}}\, d^2r$$

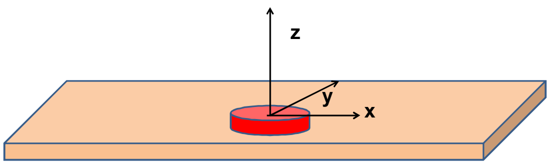
$$\widetilde{\Phi}(\mathbf{r},\omega)=\int_V \widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)f(\mathbf{r}',\omega)\,d^3r'+$$
$$\oint_S \left(\widetilde{\Phi}(\mathbf{r}',\omega)\nabla\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)-\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\widetilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}\, d^2r'$$

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Example:

$f(\mathbf{r},t) \Rightarrow$ time harmonic piston of radius a, amplitude $\varepsilon\hat{\mathbf{z}}$ can be represented as boundary value of $\Phi(\mathbf{r},t)$



Treatment of boundary values for time-harmonic force:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\tilde{f}(\mathbf{r}',\omega)d^3r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla'\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla'\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

Boundary values for our example:

$$\left(\frac{\partial\tilde{\Phi}}{\partial z}\right)_{z=0} = \begin{cases} 0 & \text{for } x^2+y^2>a^2 \\ i\omega\varepsilon a & \text{for } x^2+y^2<a^2 \end{cases}$$

Note: Need Green's function with vanishing gradient at $z=0$:

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z>0$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\,z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'\,dy'$$

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z>0$$

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)_{z'=0} = \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}; \quad z>0$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\, z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'dy'$$

$$= -i\omega\varepsilon a\int_0^a r'dr'\int_0^{2\pi} d\phi' \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}$$

Integration domain : $x' = r'\cos\phi'$

$y' = r'\sin\phi'$

For $r >> a$; $\quad |\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\phi = \frac{\pi}{2}$

$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' = r - r'\sin\theta\sin\phi'$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{i\omega\varepsilon a}{2\pi}\frac{e^{ikr}}{r}\int_0^a r'dr'\int_0^{2\pi} d\phi'\, e^{-ikr'\sin\theta\sin\phi'}$$

Note that : $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} d\phi'\, e^{-iu\sin\phi'} = J_0(u)$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a\frac{e^{ikr}}{r}\int_0^a r'dr' J_0(kr'\sin\theta)$$

$$\int_0^w u du J_0(u) = wJ_1(w)$$

$$\Rightarrow \tilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a^3\frac{e^{ikr}}{r}\frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

Energy flux : $\quad \mathbf{j}_e = \delta\mathbf{v}p$

Taking time average : $\quad \langle\mathbf{j}_e\rangle = \frac{1}{2}\Re(\delta\mathbf{v}p^*)$

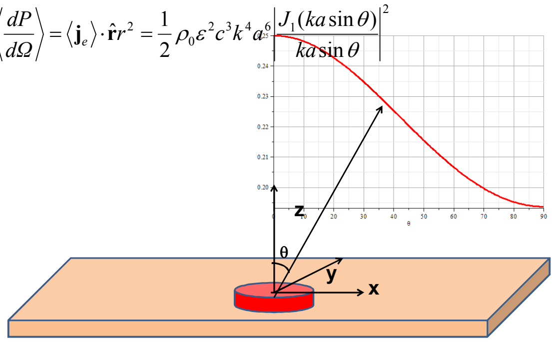
$$= \tfrac{1}{2}\rho_0\Re\left((-\nabla\Phi)(-i\omega\Phi)^*\right)$$

Time averaged power per solid angle :

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \langle\mathbf{j}_e\rangle\cdot\hat{\mathbf{r}}r^2 = \frac{1}{2}\rho_0\varepsilon^2 c^3 k^4 a^6\left|\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right|^2$$

Time averaged power per solid angle :

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2} \rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka \sin\theta)}{ka \sin\theta} \right|^2$$



Scattering of sound waves –
for example, from a rigid cylinder

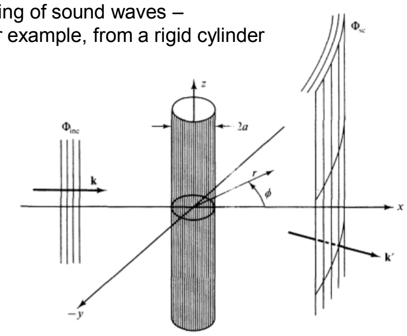


Figure 51.8 Scattering from a rigid cylinder.

Figure from Fetter and Walecka pg. 337

Scattering of sound waves –
for example, from a rigid cylinder

Velocity potential --

$$\Phi(\mathbf{r}) = \Phi_{inc}(\mathbf{r}) + \Phi_{sc}(\mathbf{r}) \qquad \Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$$

Helmholz equation in cylindrical coordinates:

$$\left(\nabla^2 + k^2\right)\Phi(\mathbf{r}) = 0 = \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \phi^2} + \frac{\partial}{\partial z^2} + k^2\right)\Phi(\mathbf{r})$$

Assume: $\Phi(\mathbf{r}) = \sum_{m=-\infty}^{\infty} e^{im\phi} R_m(r)$

where $\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + k^2\right) R_m(r) = 0$

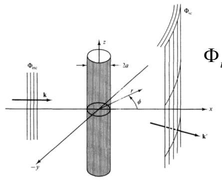


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr)$$

$$\Phi_{sc}(\mathbf{r}) = \sum_{m=-\infty}^{\infty} C_m e^{im\phi} H_m(kr) \quad \text{where Hankel function}$$

represents an outgoing wave: $H_m(kr) = J_m(kr) + iN_m(kr)$

Boundary condition at $r = a$: $\left.\dfrac{\partial \Phi}{\partial r}\right|_{r=a} = 0$

$$\Rightarrow i^m J'_m(ka) + C_m H'_m(ka) = 0 \qquad C_m = -i^m \frac{J'_m(ka)}{H'_m(ka)}$$

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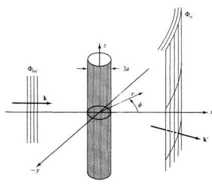

Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{sc}(\mathbf{r}) = -\sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} H_m(kr)$$

Asymptotic form:

$$i^m H_m(kr) \underset{kr\to\infty}{\approx} \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}$$

$$\Phi_{sc}(\mathbf{r}) \underset{kr\to\infty}{\approx} f(\phi)\sqrt{\frac{1}{r}} e^{ikr} - \sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} H_m(kr)$$

$$f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi-\pi/4)}$$

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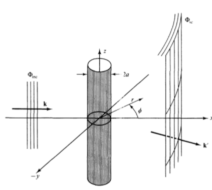
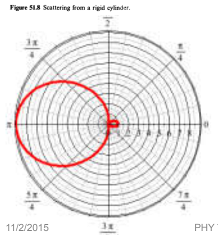


Figure 51.8 Scattering from a rigid cylinder.

$$\frac{d\sigma}{d\phi} = |f(\phi)|^2$$

For *ka << 1*

$$\frac{d\sigma}{d\phi} = |f(\phi)|^2 \approx \frac{1}{8}\pi k^3 a^4 (1 - 2\cos\phi)^2$$



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