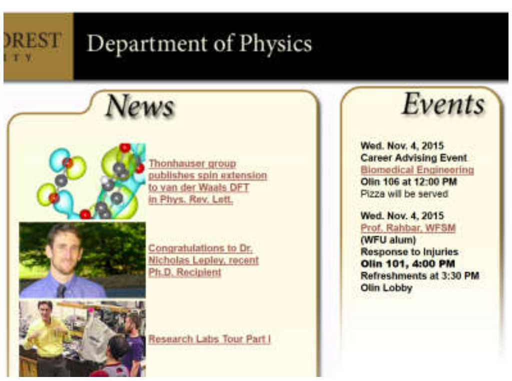
PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 28: Chap. 9 of F&W

Wave equation for sound considering both linear and non-linear effects

1. **Sound scattering**
2. **Non-linear effects in traveling sound waves**
3. **Shocking effects**

14	Fri, 9/25/2015	Chap. 4	Small oscillations	#14
15	Mon, 9/28/2015	Chap. 4	Normal modes of motion	#15
16	Wed, 9/30/2015	Chap. 7	Wave motion	#16
17	Fri, 10/02/2015	Chap. 7 & App. A	Contour Integration	#17
18	Mon, 10/05/2015	Chap. 7	Fourier transforms	#18
19	Wed, 10/07/2015	Chap. 7	Laplace transforms	#19
20	Fri, 10/09/2015	Chap. 7	Green's functions	Start exam
	Mon, 10/12/2015		No class	Take home exam
	Wed, 10/14/2015		No class	Exam due before 10/19/2015
	Fri, 10/16/2015		Fall break -- no class	
21	Mon, 10/19/2015	Chap. 5	Motion of Rigid Bodies	#20
22	Wed, 10/21/2015	Chap. 5	Motion of Rigid Bodies	#21
23	Fri, 10/23/2015	Chap. 8	Motion of Elastic membranes	#22
24	Mon, 10/26/2015	Chap. 9	Hydrodynamics	#23
25	Wed, 10/28/2015	Chap. 9	Hydrodynamics	#24
26	Fri, 10/30/2015	Chap. 9	Sound waves	#25
27	Mon, 11/02/2015	Chap. 9	Sound waves	#26
28	Wed, 11/04/2015	Chap. 9	Sound waves	
	Wed, 12/02/2015		Student presentations I	
	Fri, 12/04/2015		Student presentations II	
	Mon, 12/07/2015		Begin Take-home final	



WFU Physics Colloquium

TITLE: The response to traumatic injuries: A journey down the road less traveled

SPEAKER: Professor Elaheh Rahbar, (WFU alum)

Biomedical Engineering Center for Public Health Genomics, Wake Forest School of Medicine

TIME: Wednesday November 4, 2015 at 4:00 PM

PLACE: Room 101 Olin Physical Laboratory

Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

ABSTRACT

Trauma is the leading cause of death in people under the age of 45 years in both civilian and military populations, with hemorrhagic shock accounting for nearly 50% of these fatalities. Death from hemorrhagic shock is considered potentially preventable with appropriate resuscitation. However there remains no optimal or standardized resuscitation protocol for trauma patients. In this seminar, I will present findings from various clinical prospective observational studies, pilot clinical trials and the most recent findings from the multi-site randomized clinical trial PROPPR; all which investigated the physiologic response

Solutions to linear wave equation in terms of velocity potential:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

Plane wave solution :

$$\Phi(\mathbf{r},t) = Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$

Scattering of sound waves –

for example, from a rigid cylinder

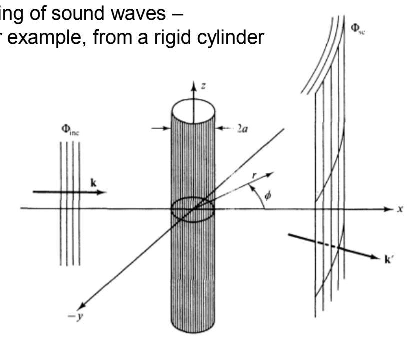


Figure 51.8 Scattering from a rigid cylinder.

Figure from Fetter and Walecka pg. 337

Scattering of sound waves –
for example, from a rigid cylinder

Velocity potential --

$$\Phi(\mathbf{r}) = \Phi_{inc}(\mathbf{r}) + \Phi_{sc}(\mathbf{r}) \qquad \Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$$

Helmholz equation in cylindrical coordinates:

$$\left(\nabla^2 + k^2\right)\Phi(\mathbf{r}) = 0 = \left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\Phi(\mathbf{r})$$

Assume: $\Phi(\mathbf{r}) = \sum_{m=-\infty}^{\infty} e^{im\phi} R_m(r)$ (no *z* dependence)

where $\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} + k^2\right) R_m(r) = 0$

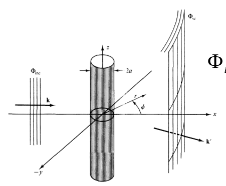


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{inc}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr)$$

$\Phi_{sc}(\mathbf{r}) = \sum_{m=-\infty}^{\infty} C_m e^{im\phi} H_m(kr)$ where Hankel function

represents an outgoing wave : $H_m(kr) = J_m(kr) + iN_m(kr)$

Boundary condition at $r = a$: $\left.\frac{\partial\Phi}{\partial r}\right|_{r=a} = 0$

$$\Rightarrow i^m J'_m(ka) + C_m H'_m(ka) = 0 \qquad C_m = -i^m \frac{J'_m(ka)}{H'_m(ka)}$$

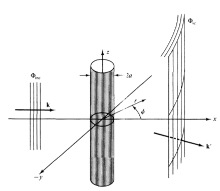


Figure 51.8 Scattering from a rigid cylinder.

$$\Phi_{sc}(\mathbf{r}) = -\sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} H_m(kr)$$

Asymptotic form:

$$i^m H_m(kr) \underset{kr\to\infty}{\approx} \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}$$

$$\Phi_{sc}(\mathbf{r}) \underset{kr\to\infty}{\approx} f(\phi)\sqrt{\frac{1}{r}} e^{ikr} = -\sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka)}{H'_m(ka)} e^{im\phi} H_m(kr)$$

$$f(\phi) = -\sqrt{\frac{2}{\pi k}} \sum_{m=-\infty}^{\infty} \frac{J'_m(ka)}{H'_m(ka)} e^{i(m\phi-\pi/4)}$$

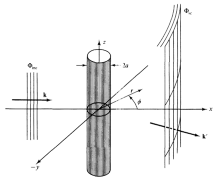
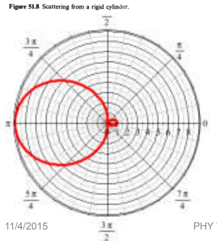


Figure 51.8 Scattering from a rigid cylinder.



$$\frac{d\sigma}{d\phi} = \left|f(\phi)\right|^2$$

For *ka << 1*

$$\frac{d\sigma}{d\phi} = \left|f(\phi)\right|^2 \approx \frac{1}{8}\pi k^3 a^4 \left(1 - 2\cos\phi\right)^2$$

Now consider non-linear effects in sound

Effects of nonlinearities in fluid equations
-- one dimensional case

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}\cdot\nabla\right)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v}\right) = 0$

Assume spatial variation confined to x direction ;
assume that $\mathbf{v} = v\hat{\mathbf{x}}$ and $\mathbf{f}_{applied} = 0$.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0$$

$$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}+\rho\frac{\partial v}{\partial x}=0$$

Expressing p in terms of ρ: $p=p(\rho)$

$$\frac{\partial p}{\partial x}=\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x}\equiv c^2(\rho)\frac{\partial \rho}{\partial x} \qquad \text{where } \frac{\partial p}{\partial \rho}\equiv c^2(\rho)$$

For adiabatic ideal gas: $\frac{dp}{p}=\gamma\frac{d\rho}{\rho} \qquad p=p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}$

$$c^2(\rho)=\frac{\gamma p}{\rho}=c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \qquad \text{where } c_0^2\equiv\frac{\gamma p_0}{\rho_0}$$

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$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$

$$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}+\rho\frac{\partial v}{\partial x}=0$$

Expressing variation of v in terms of $v(\rho)$:

$$\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial t}+v\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$

$$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}+\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}=0$$

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Some more algebra :

From Euler equation : $\frac{\partial v}{\partial \rho}\left(\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$

From continuity equation : $\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}=-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}$

Combined equation : $\frac{\partial v}{\partial \rho}\left(-\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}\right)+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$

$$\Rightarrow \left(\frac{\partial v}{\partial \rho}\right)^2=\frac{c^2(\rho)}{\rho^2} \qquad \frac{\partial v}{\partial \rho}=\pm\frac{c}{\rho}$$

$$\Rightarrow \frac{\partial \rho}{\partial t}+(v\pm c)\frac{\partial \rho}{\partial x}=0$$

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Assuming adiabatic process : $c^2 = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ $\qquad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$\frac{\partial v}{\partial \rho} = \frac{dv}{d\rho} = \pm\frac{c}{\rho} \qquad \Rightarrow v = \pm c_0 \int_{\rho_0}^{\rho}\left(\frac{\rho'}{\rho_0}\right)^{(\gamma-1)/2}\frac{d\rho'}{\rho'}$$

$$\Rightarrow v = \pm\frac{2c_0}{\gamma-1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right)$$

$$\Rightarrow c = c_0\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2}$$

Summary :

$$\frac{dv}{d\rho} = \pm\frac{c}{\rho}$$

$$\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$$

Assuming adiabatic process : $c^2 = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$ $\qquad c_0^2 = \frac{\gamma p_0}{\rho_0}$

$$c = c_0\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} \qquad v = \pm\frac{2c_0}{\gamma-1}\left(\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - 1\right)$$

Traveling wave solution:

Assume : $\rho = \rho_0 + f(x - u(\rho)t)$

Need to find self - consistent equations for propagation velocity $u(\rho)$ using equations

From previous derivations : $\frac{\partial \rho}{\partial t} + (v \pm c)\frac{\partial \rho}{\partial x} = 0$

Apparently : $u(\rho) \Leftrightarrow v \pm c$

For adiabatic ideal gas and + signs :

$$u = v + c = c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2} - \frac{2}{\gamma-1}\right)$$

Traveling wave solution -- continued:

$$\frac{\partial\rho}{\partial t}+(v\pm c)\frac{\partial\rho}{\partial x}=0$$

Assume: $\rho=\rho_0+f(x-u(\rho)t)=\rho_0+f(x-(v\pm c)t)$

For adiabatic ideal gas and + signs:

$$u=v+c=c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2}-\frac{2}{\gamma-1}\right)$$

Solution in linear approxiation:

$$u=v+c\approx v_0+c_0=c_0\left(\frac{\gamma+1}{\gamma-1}-\frac{2}{\gamma-1}\right)=c_0$$

$$\Rightarrow\rho=\rho_0+f(x-c_0t)$$

Traveling wave solution -- full non-linear case:

Visualization for particular waveform: $\rho=\rho_0+f(\underbrace{x-u(\rho)t}_{w})$

Assume: $f(w)\equiv f_0 s(w)$

$$\frac{\rho}{\rho_0}=1+\frac{f_0}{\rho_0}s(x-ut)$$

For adiabatic ideal gas:

$$u=c_0\left(\frac{\gamma+1}{\gamma-1}\left(\frac{\rho}{\rho_0}\right)^{(\gamma-1)/2}-\frac{2}{\gamma-1}\right)$$

$$u=c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+\frac{f_0}{\rho_0}s(x-ut)\right)^{(\gamma-1)/2}-\frac{2}{\gamma-1}\right)$$

Visualization continued:

$$u=c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+\frac{f_0}{\rho_0}s(x-ut)\right)^{(\gamma-1)/2}-\frac{2}{\gamma-1}\right)$$

Plot $s(x-ut)$ for fixed t, as a function of x:

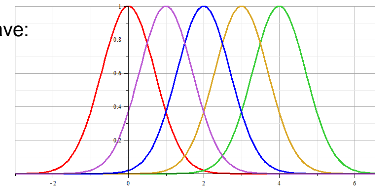
Let $w=x-ut$

$$x=w+ut=w+u(w)t\equiv x(w,t)$$

$$u(w)=c_0\left(\frac{\gamma+1}{\gamma-1}\left(1+\frac{f_0}{\rho_0}s(w)\right)^{(\gamma-1)/2}-\frac{2}{\gamma-1}\right)$$

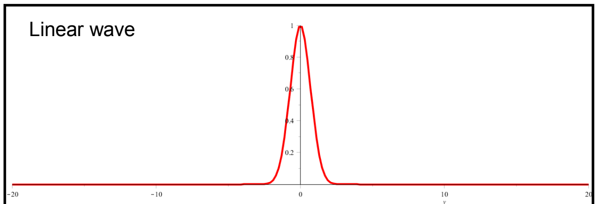
Parametric equations: plot $s(w)$ vs $x(w,t)$ for range of w

Linear wave:

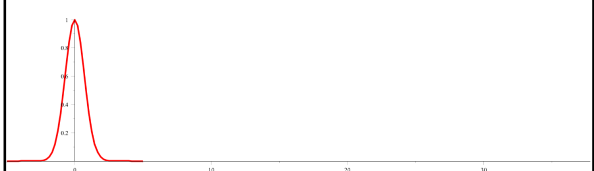


Non-linear wave:

Linear wave

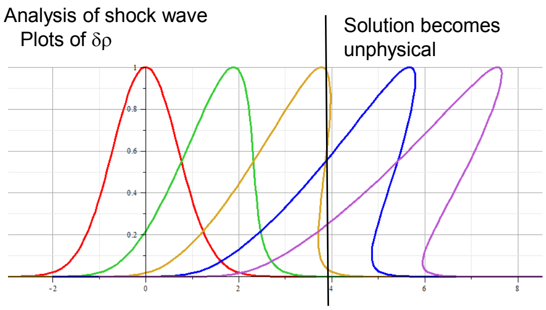


Non-linear wave

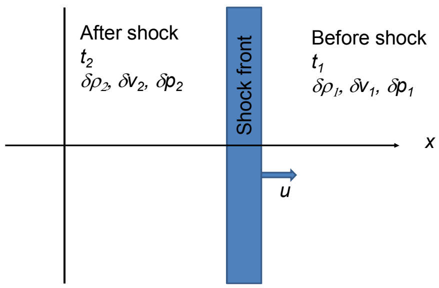


Analysis of shock wave
Plots of $\delta\rho$

Solution becomes unphysical



Analysis of shock wave -- continued



Analysis of shock wave – continued

While analysis in the shock region is complicated, we can use conservation laws to analyze regions 1 and 2

Assume $\rho(x,t)=\rho(x-ut)$

$p(x,t)=p(x-ut)$

$v(x,t)=v(x-ut)$

Continuity equation:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0=\frac{\partial(\rho v-\rho u)}{\partial x} \qquad \Rightarrow (v_2-u)\rho_2=(v_1-u)\rho_1$$

Conservation of energy and momentum:

$$\Rightarrow p_2+\rho_2(v_2-u)^2=p_1+\rho_1(v_1-u)^2$$

$$\Rightarrow \epsilon_2+\frac{1}{2}(v_2-u)^2+\frac{p_2}{\rho_2}=\epsilon_1+\frac{1}{2}(v_1-u)^2+\frac{p_1}{\rho_1}$$

Analysis of shock wave – continued

For adiabatic ideal gas, also considering energy and momentum conservation:

$$\frac{\rho_2}{\rho_1}=\frac{\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1}+1}{\frac{\gamma+1}{\gamma-1}+\frac{p_2}{p_1}}\le\frac{\gamma+1}{\gamma-1}$$

Velocity relationships:

$$\frac{(v_1-u)^2}{c_1^2}=\frac{1}{2\gamma}\left(\gamma-1+(\gamma+1)\frac{p_2}{p_1}\right) \qquad \frac{(v_2-u)^2}{c_2^2}=\frac{1}{2\gamma}\left(\gamma-1+(\gamma+1)\frac{p_1}{p_2}\right)$$

where $c_1^2\equiv\frac{\gamma p_1}{\rho_1}$ and $c_2^2\equiv\frac{\gamma p_2}{\rho_2}$

For a strong shock:

$$\frac{(v_1-u)^2}{c_1^2}\to\frac{(\gamma+1)}{2\gamma}\frac{p_2}{p_1} \qquad \frac{(v_2-u)^2}{c_2^2}\to\frac{(\gamma-1)}{2\gamma}$$

Analysis of shock wave – continued

For adiabatic ideal gas, entropy considerations::

Internal energy density: $\varepsilon = \frac{p}{(\gamma - 1)\rho} = C_V T$

First law of thermo: $d\varepsilon = Tds - pd\left(\frac{1}{\rho}\right)$

$$ds = \frac{1}{T}\left(d\left(\frac{p}{(\gamma-1)\rho}\right) - pd\left(\frac{1}{\rho}\right)\right) = C_V d \ln\left(\frac{p}{\rho^\gamma}\right)$$

$$s = C_V \ln\left(\frac{p}{\rho^\gamma}\right) + (\text{constant})$$

$$s_2 - s_1 = C_V \ln\left(\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^\gamma\right) \qquad 0 < s_2 - s_1 < C_V\left(\ln\left(\frac{p_2}{p_1}\right) - \gamma \ln\left(\frac{\gamma+1}{\gamma-1}\right)\right)$$