PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 33:

Chapter 10 in F & W: Surface waves

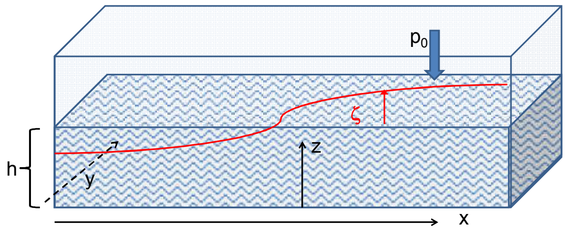
1. **Water waves in a channel**
2. **Wave-like solutions; wave speed**

15	Mon, 9/28/2015	Chap. 4	Normal modes of motion	#15
16	Wed, 9/30/2015	Chap. 7	Wave motion	#16
17	Fri, 10/02/2015	Chap. 7 & App. A	Contour Integration	#17
18	Mon, 10/05/2015	Chap. 7	Fourier transforms	#18
19	Wed, 10/07/2015	Chap. 7	Laplace transforms	#19
20	Fri, 10/09/2015	Chap. 7	Green's functions	Start exam
	Mon, 10/12/2015		No class	Take home exam
	Wed, 10/14/2015		No class	Exam due before 10/19/2015
	Fri, 10/16/2015		Fall break -- no class	
21	Mon, 10/19/2015	Chap. 5	Motion of Rigid Bodies	#20
22	Wed, 10/21/2015	Chap. 5	Motion of Rigid Bodies	#21
23	Fri, 10/23/2015	Chap. 8	Motion of Elastic membranes	#22
24	Mon, 10/26/2015	Chap. 9	Hydrodynamics	#23
25	Wed, 10/28/2015	Chap. 9	Hydrodynamics	#24
26	Fri, 10/30/2015	Chap. 9	Sound waves	#25
27	Mon, 11/02/2015	Chap. 9	Sound waves	#26
28	Wed, 11/04/2015	Chap. 9	Sound waves	
29	Fri, 11/06/2015	Chap. 10	Surface waves on fluids	#27
	Wed, 12/02/2015		Student presentations I	
	Fri, 12/04/2015		Student presentations II	
	Mon, 12/07/2015		Begin Take-home final	

Physics of incompressible fluids and their surfaces

Reference: Chapter 10 of Fetter and Walecka

Consider a container of water with average height h and surface h+ζ(x,y,t); (h ←→ z_0 on some of the slides)



Euler's equation for incompressible fluid :

$$\frac{d\mathbf{v}}{dt} = f_{applied} - \frac{\nabla p}{\rho} = -g\hat{\mathbf{z}} - \frac{\nabla p}{\rho}$$

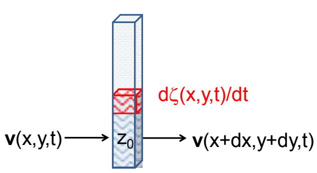
Assume that $v_z << v_x, v_y$ $\Rightarrow -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \approx 0$

$$\Rightarrow p(x,y,z,t) = p_0 + \rho g\left(\zeta(x,y,t) + h - z\right)$$

Horizontal fluid motions :

$$\frac{dv_x}{dt} \approx \frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial \zeta}{\partial x}$$

$$\frac{dv_y}{dt} \approx \frac{\partial v_y}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial \zeta}{\partial y}$$



Continuity condition for flow of incompressible fluid :

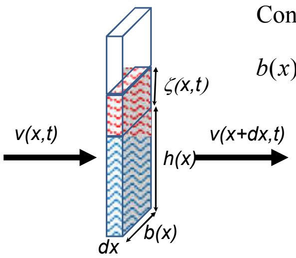
$$\frac{\partial \zeta}{\partial t} + z_0 \nabla \cdot \mathbf{v} = 0$$

From horizontal flow relations : $\frac{\partial \mathbf{v}}{\partial t} = -g\nabla\zeta$

Equation for surface function : $\frac{\partial^2 \zeta}{\partial t^2} - g z_0 \nabla^2 \zeta = 0$

Digression:

The form of the continuity equation given on previous (and subsequent) sides assumes that the transverse cross section of the channel is either very large or at least uniform. Your text also considers the case where there is a channel of varying width:



Continuity condition:

$$b(x)\frac{\partial \zeta}{\partial t} = -\frac{\partial}{\partial x}\left(h(x)b(x)v(x,t)\right)$$

For constant b and h:

$$\frac{\partial \zeta}{\partial t} = -h\frac{\partial}{\partial x}\left(v(x,t)\right)$$

For uniform channel:

Surface wave equation:

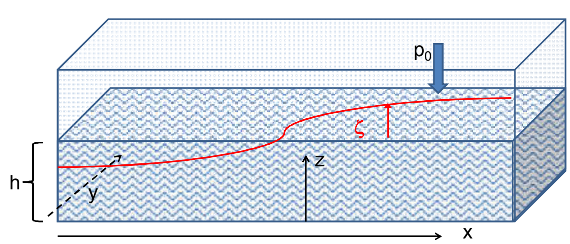
$$\frac{\partial^2 \zeta}{\partial t^2} - c^2\nabla^2\zeta = 0 \qquad c^2 = gh$$

More complete analysis finds:

$$c^2 = \frac{g}{k}\tanh(kh) \qquad \text{where } k = \frac{2\pi}{\lambda}$$

More details: -- recall setup --

Consider a container of water with average height h and surface h+ζ(x,y,t)



Equations describing fluid itself (without boundaries)

Euler's equation for incompressible fluid:

$$\frac{d\mathbf{v}}{dt}=\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}=\frac{\partial \mathbf{v}}{\partial t}+\nabla\left(\tfrac{1}{2}v^2\right)+\mathbf{v}\times\left(\nabla\times\mathbf{v}\right)=-\nabla U-\frac{\nabla p}{\rho}$$

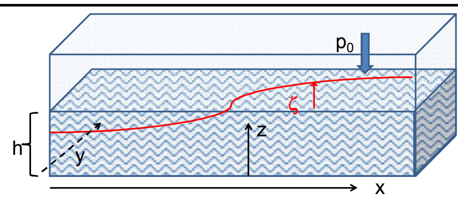
Assume that $\nabla\times\mathbf{v}=0$ (irrotational flow) $\Rightarrow \mathbf{v}=-\nabla\Phi$

$$\Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}v^2+U+\frac{p}{\rho}\right)=0$$

$$\Rightarrow -\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}v^2+U+\frac{p}{\rho}=\text{constant (within the fluid)}$$

For the same system, the continuity condition becomes

$$\nabla\cdot\mathbf{v}=-\nabla^2\Phi=0$$



Within fluid: $0\le z\le h+\zeta$

$$-\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}v^2+g\left(z-h\right)=\text{constant}$$

(We have absorbed p_0 in "constant")

$$-\nabla^2\Phi=0$$

At surface: $z=h+\zeta$ with $\zeta=\zeta\left(x,y,t\right)$

$$\frac{d\zeta}{dt}=\frac{\partial\zeta}{\partial t}+v_x\frac{\partial\zeta}{\partial x}+v_y\frac{\partial\zeta}{\partial y}\quad \text{where } v_{x,y}=v_{x,y}\left(x,y,h+\zeta,t\right)$$

Full equations:

Within fluid: $0\le z\le h+\zeta$

$$-\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}v^2+g\left(z-h\right)=\text{constant}$$

(We have absorbed p_0 in "constant")

$$-\nabla^2\Phi=0$$

At surface: $z=h+\zeta$ with $\zeta=\zeta\left(x,y,t\right)$

$$\frac{d\zeta}{dt}=\frac{\partial\zeta}{\partial t}+v_x\frac{\partial\zeta}{\partial x}+v_y\frac{\partial\zeta}{\partial y}\quad \text{where } v_{x,y}=v_{x,y}\left(x,y,h+\zeta,t\right)$$

Linearized equations:

For $0\le z\le h+\zeta$: $-\dfrac{\partial\Phi}{\partial t}+g(z-h)=0 \qquad -\nabla^2\Phi=0$

At surface: $z=h+\zeta \qquad \dfrac{d\zeta}{dt}=\dfrac{\partial\zeta}{\partial t}=v_z(x,y,h+\zeta,t)$

$$-\frac{\partial\Phi(x,y,h+\zeta,t)}{\partial t}+g\zeta=0$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x*:

$$\text{For} \quad 0 \le z \le h+\zeta: \qquad \nabla^2\Phi = \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2}\right)\Phi(x,z,t) = 0$$

$$\text{Consider and periodic waveform:} \quad \Phi(x,z,t) = Z(z)\cos\big(k(x-ct)\big)$$

$$\Rightarrow \left(\frac{d^2}{dz^2} - k^2\right) Z(z) = 0$$

$$\text{Boundary condition at bottom of tank:} \quad v_z(x,0,t) = 0$$

$$\Rightarrow \frac{dZ}{dz}(0) = 0 \qquad\qquad Z(z) = A\cosh(kz)$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$\text{At surface:} \quad z = h+\zeta \qquad \frac{\partial \zeta}{\partial t} = v_z(x,h+\zeta,t) = -\frac{\partial \Phi(x,h+\zeta,t)}{\partial z}$$

$$-\frac{\partial \Phi(x,h+\zeta,t)}{\partial t} + g\zeta = 0$$

$$-\frac{\partial^2 \Phi(x,h+\zeta,t)}{\partial t^2} + g\frac{\partial \zeta}{\partial t} = -\frac{\partial^2 \Phi(x,h+\zeta,t)}{\partial t^2} - g\frac{\partial \Phi(x,h+\zeta,t)}{\partial z} = 0$$

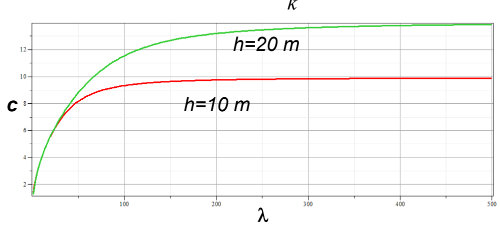
$$\text{For} \quad \Phi(x,(h+\zeta),t) = A\cosh(k(h+\zeta))\cos(k(x-ct))$$

$$A\cosh(k(h+\zeta))\cos(k(x-ct))\left(k^2c^2 - gk\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))}\right) = 0$$

For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 = \frac{g}{k}\frac{\sinh(k(h+\zeta))}{\cosh(k(h+\zeta))} = \frac{g}{k}\tanh(k(h+\zeta))$$

$$\text{Assuming } \zeta << h: \qquad c^2 = \frac{g}{k}\tanh(kh)$$



For simplicity, keep only linear terms and assume that horizontal variation is only along *x* – continued:

$$c^2 \approx \frac{g}{k}\tanh(kh) \qquad \text{For } \lambda >> h,\ c^2 \approx gh$$

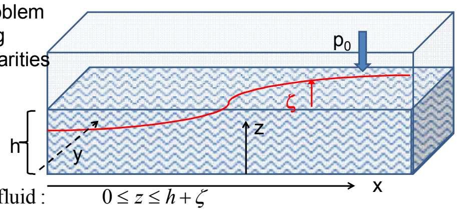
$$\Phi(x,z,t) = A\cosh(kz)\cos(k(x-ct))$$

$$\zeta(x,t) = \frac{1}{g}\frac{\partial\Phi(x,h+\zeta,t)}{\partial t} \approx \frac{kc}{g}A\cosh(kh)\sin(k(x-ct))$$

Note that for $\lambda >> h, \quad c^2 \approx gh$

(solutions are consistent with previous analysis)

General problem including non-linearities



Within fluid : $\quad 0 \le z \le h+\zeta$

$$-\frac{\partial\Phi}{\partial t} + \tfrac{1}{2}v^2 + g(z-h) = \text{constant} \quad \text{(We have absorbed}$$

$$-\nabla^2\Phi = 0 \qquad\qquad p_0 \text{ in our constant.)}$$

At surface : $\quad z = h+\zeta \qquad$ with $\zeta = \zeta(x,y,t)$

$$\frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} + v_x\frac{\partial\zeta}{\partial x} + v_y\frac{\partial\zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}(x,y,h+\zeta,t)$$