

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF Olin 103**

Plan for Lecture 2:

1. Brief comment on quiz
2. Particle interactions
3. Notion of center of mass
reference frame
4. Introduction to scattering theory

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Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/26/2015	Chap. 1	Review of basic principles	#1
2	Fri, 8/28/2015	Chap. 1	Scattering theory	#2
3	Mon, 8/31/2015	Chap. 1	Scattering theory continued	
4	Wed, 9/02/2015	Chap. 2	Accelerated coordinate systems	
5	Fri, 9/04/2015	Chap. 3	Calculus of variations	
6	Mon, 9/07/2015	Chap. 3	Calculus of variations	
7	Wed, 9/09/2015	Chap. 3	Hamilton's principle	
8	Fri, 9/11/2015	Chap. 3 & 6	Hamilton's principle	
9	Mon, 9/14/2015	Chap. 3 & 6	Lagrangians with constraints	
10	Wed, 9/16/2015	Chap. 3 & 6	Lagrangians and constants of motion	
11	Fri, 9/18/2015	Chap. 3 & 6	Hamiltonian formalism	
12	Mon, 9/21/2015	Chap. 3 & 6	Hamiltonian formalism	
13	Wed, 9/23/2015	Chap. 3 & 6	Hamiltonian Jacobi transformations	

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Comment on quiz questions

1. $g(t) = \int_0^t (x^2 + t) dx \quad \frac{dg}{dt} = \int_0^t \frac{d(x^2 + t)}{dt} + (x^2 + t) \Big|_{x=t} = t^2 + 2t$

2. Evaluate the integral $\oint \frac{dz}{z}$ for a closed contour about the origin.

$$\oint \frac{dz}{z} = 2\pi i$$

3. $\frac{df}{dx} = f \quad \Rightarrow f(x) = Ae^x$

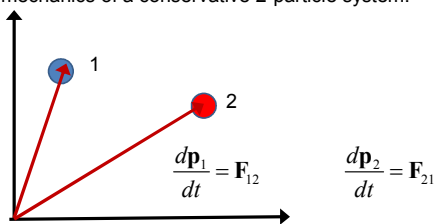
4. $\sum_{n=1}^N a^n = \frac{a - a^{N+1}}{1 - a}$

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Classical mechanics of a conservative 2-particle system.



$$\mathbf{F}_{12} = -\nabla_1 V(\mathbf{r}_1 - \mathbf{r}_2) \Rightarrow E = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

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Relationship between center of mass and laboratory frames of reference

Definition of center of mass $\mathbf{R}_{CM}$

$$m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 = (m_1 + m_2) \mathbf{R}_{CM}$$

$$m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2 = (m_1 + m_2) \dot{\mathbf{R}}_{CM} = (m_1 + m_2) \mathbf{V}_{CM}$$

$$E = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

$$= \frac{1}{2} (m_1 + m_2) V_{CM}^2 + \frac{1}{2} \mu |\mathbf{v}_1 - \mathbf{v}_2|^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

where: $\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$

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Classical mechanics of a conservative 2-particle system -- continued

$$E = \frac{1}{2} (m_1 + m_2) V_{CM}^2 + \frac{1}{2} \mu |\mathbf{v}_1 - \mathbf{v}_2|^2 + V(\mathbf{r}_1 - \mathbf{r}_2)$$

For central potentials: $V(\mathbf{r}_1 - \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|) \equiv V(r_{12})$

Relative angular momentum is also conserved:

$$\mathbf{L}_{12} \equiv \mathbf{r}_{12} \times \mu \mathbf{v}_{12}$$

$$E = \frac{1}{2} (m_1 + m_2) V_{CM}^2 + \frac{1}{2} \mu v_{12}^2 + \frac{L_{12}^2}{2\mu r_{12}^2} + V(r_{12})$$

Simpler notation:

$$E = \frac{1}{2} (m_1 + m_2) V_{CM}^2 + \frac{1}{2} \mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

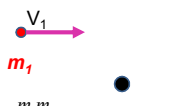
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Note: The following analysis will be carried out in the center of mass frame of reference.

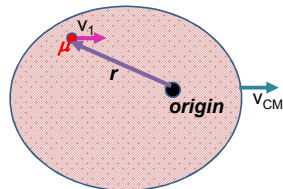
In laboratory frame:



$$\mu = \frac{m_1 m_{\text{target}}}{m_1 + m_{\text{target}}}$$

$$\ell = |\mathbf{r} \times \mu \mathbf{v}_1|$$

In center-of-mass frame:



Also note: We are assuming that the interaction between particle and target $V(r)$ conserves energy and angular momentum.

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Scattering theory:

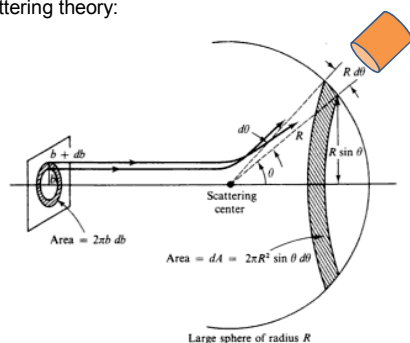


Figure 5.5 The scattering problem and relation of cross section to impact parameter.

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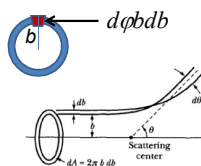
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Differential cross section

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

= Area of incident beam that is scattered into detector at angle θ



$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{d\phi b db}{d\phi \sin \theta d\theta} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right|$$

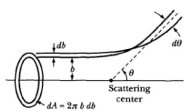
Figure from Marion & Thorton, Classical Dynamics

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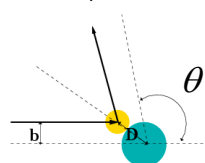
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Simple example – collision of hard spheres



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta} \left|\frac{db}{d\theta}\right|$$

Microscopic view:



$$b(\theta) = ?$$

$$b(\theta) = D \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{D^2}{4}$$

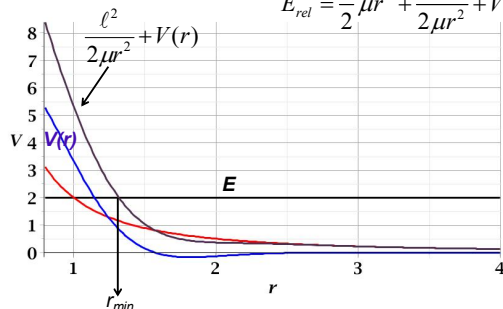
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For a continuous potential interaction' in center of mass reference frame:

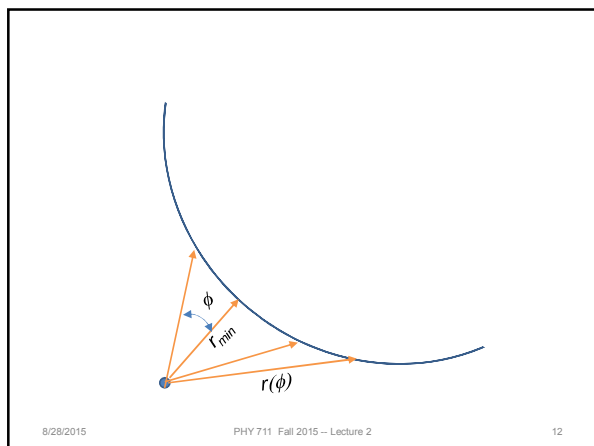
$$E_{rel} = \frac{1}{2} \mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$



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Conservation of angular momentum:

$$\ell = \mu r^2 \left(\frac{d\phi}{dt} \right)$$

Transformation of trajectory variables:

$$r(t) \Leftrightarrow r(\phi)$$

$$\frac{dr}{dt} = \frac{dr}{d\phi} \frac{d\phi}{dt} = \frac{dr}{d\phi} \frac{\ell}{\mu r^2}$$

Conservation of energy in the center of mass frame:

$$E = \frac{1}{2} \mu \left(\frac{dr}{dt} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$= \frac{1}{2} \mu \left(\frac{dr}{d\phi} \frac{\ell}{\mu r^2} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

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$$\Rightarrow E = \frac{1}{2} \mu \left(\frac{dr}{dt} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

$$= \frac{1}{2} \mu \left(\frac{dr}{d\phi} \frac{\ell}{\mu r^2} \right)^2 + \frac{\ell^2}{2\mu r^2} + V(r)$$

Solving for $r(\phi) \Leftrightarrow \phi(r)$

$$\left(\frac{dr}{d\phi} \right)^2 = \left(\frac{2\mu r^4}{\ell^2} \right) \left(E - \frac{\ell^2}{2\mu r^2} - V(r) \right)$$

$$d\phi = dr \left(\frac{\ell / r^2}{\sqrt{2\mu \left(E - \frac{\ell^2}{2\mu r^2} - V(r) \right)}} \right)$$

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$$d\phi = dr \left(\frac{\ell / r^2}{\sqrt{2\mu \left(E - \frac{\ell^2}{2\mu r^2} - V(r) \right)}} \right)$$

Further simplification at large separation:

$$\ell = \mu v_{\infty} b$$

$$E = \frac{1}{2} \mu v_{\infty}^2$$

$$\Rightarrow \ell = \sqrt{2\mu E} b$$

$$d\phi = dr \left(\frac{\sqrt{2\mu E} b / r^2}{\sqrt{2\mu \left(E - \frac{2\mu E b^2}{2\mu r^2} - V(r) \right)}} \right)$$

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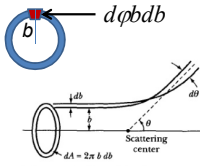
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Differential cross section

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{\text{Number of detected particles at } \theta \text{ per target particle}}{\text{Number of incident particles per unit area}}$$

= Area of incident beam that is scattered into detector at angle θ



$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{d\phi b db}{d\phi \sin \theta d\theta} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right|$$

Figure from Marion & Thorton, Classical Dynamics

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When the dust clears :

$$d\phi = dr \left(\frac{\ell / r^2}{\sqrt{2\mu \left(E - \frac{\ell^2}{2\mu r^2} - V(r) \right)}} \right)$$

$$d\phi = dr \left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\Rightarrow \phi(b, E)$$

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$$\int_0^{\phi_{\max}} d\phi = \int_{r_{\min}}^{\infty} dr \left(\frac{b / r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

where :

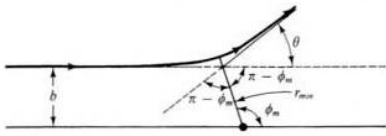
$$1 - \frac{b^2}{r_{\min}^2} - \frac{V(r_{\min})}{E} = 0$$

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Relationship between $\phi_{\max}$ and θ :



$$2(\pi - \phi_{\max}) + \theta = \pi$$

$$\Rightarrow \phi_{\max} = \frac{\pi}{2} - \frac{\theta}{2}$$

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$$\phi_{\max} = \frac{\pi}{2} - \frac{\theta}{2} = \int_{r_{\min}}^{\infty} dr \left(\frac{b/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = \pi - 2b \int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1 - \frac{b^2}{r^2} - \frac{V(r)}{E}}} \right)$$

$$\theta = \pi - 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \frac{V(1/u)}{E}}} \right)$$

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Scattering angle equation :

$$\theta = \pi - 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \frac{V(1/u)}{E}}} \right)$$

Rutherford scattering example :

$$\frac{V(r)}{E} \equiv \frac{\kappa}{r} \quad \frac{1}{r_{\min}} = \frac{1}{b} \left(-\frac{\kappa}{2b} + \sqrt{\left(\frac{\kappa}{2b}\right)^2 + 1} \right)$$

$$\theta = \pi - 2b \int_0^{1/r_{\min}} du \left(\frac{1}{\sqrt{1 - b^2 u^2 - \kappa u}} \right) = 2 \sin^{-1} \left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

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Rutherford scattering continued :

$$\theta = 2 \sin^{-1} \left(\frac{1}{\sqrt{(2b/\kappa)^2 + 1}} \right)$$

$$\frac{2b}{\kappa} = \left| \frac{\cos(\theta/2)}{\sin(\theta/2)} \right|$$

$$\left(\frac{d\sigma}{d\Omega} \right) = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right| = \frac{\kappa^2}{16} \frac{1}{\sin^4(\theta/2)}$$

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