

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF Olin 103**

Plan for Lecture 30:

**Chapter 10 in F & W: Surface waves
-- Non-linear contributions and soliton
solutions**

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22 Wed, 10/21/2015	Chap. 5	Motion of Rigid Bodies	#21
23 Fri, 10/23/2015	Chap. 8	Motion of Elastic membranes	#22
24 Mon, 10/26/2015	Chap. 9	Hydrodynamics	#23
25 Wed, 10/28/2015	Chap. 9	Hydrodynamics	#24
26 Fri, 10/30/2015	Chap. 9	Sound waves	#25
27 Mon, 11/02/2015	Chap. 9	Sound waves	#26
28 Wed, 11/04/2015	Chap. 9	Sound waves	
29 Fri, 11/06/2015	Chap. 10	Surface waves on fluids	#27
30 Mon, 11/09/2015	Chap. 10	Surface waves on fluids	#28
31 Wed, 11/11/2015	Chap. 11	Heat Conduction	
32 Fri, 11/13/2015	Chap. 11	Heat Conduction	
33 Mon, 11/16/2015	Chap. 11	Heat Conduction	
34 Wed, 11/18/2015	Chap. 12	Viscosity	
35 Fri, 11/20/2015	Chap. 12	More viscosity	
36 Mon, 11/23/2015	Chap. 13	Elastic Continua	
Wed, 11/25/2015		Thanksgiving Holiday	
Fri, 11/27/2015		Thanksgiving Holiday	
37 Mon, 11/30/2015	Chap. 13	Elastic Continua	
Wed, 12/02/2015		Student presentations I	
Fri, 12/04/2015		Student presentations II	
Mon, 12/07/2015		Begin Take-home final	

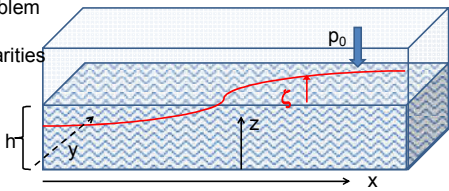
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Surface waves in an incompressible fluid

General problem including non-linearities



Within fluid: $0 \leq z \leq h + \zeta$

$$-\frac{\partial \Phi}{\partial t} + \frac{1}{2} v^2 + g(z - h) = \text{constant} \quad \Phi = \Phi(x, y, z, t)$$

$$-\nabla^2 \Phi = 0 \quad \mathbf{v} = \mathbf{v}(x, y, z, t) = -\nabla \Phi(x, y, z, t)$$

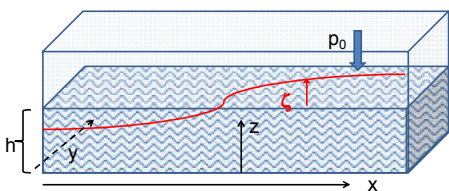
At surface: $z = h + \zeta$ with $\zeta = \zeta(x, y, t)$

$$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_x \frac{\partial \zeta}{\partial x} + v_y \frac{\partial \zeta}{\partial y} \quad \text{where } v_{x,y} = v_{x,y}(x, y, h + \zeta, t)$$

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Further simplifications; assume trivial y -dependence

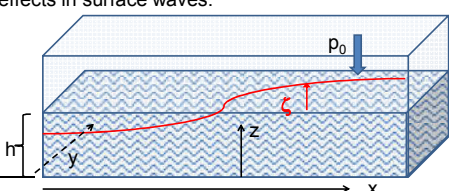
$$\Phi = \Phi(x, z, t) \quad \zeta = \zeta(x, t)$$

Within fluid: $0 \leq z \leq h + \zeta$

At surface: $v_z(x, z = h + \zeta, t) = -\frac{\partial \Phi}{\partial z} = \frac{d\zeta}{dt}$

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Non-linear effects in surface waves:



Dominant non-linear effects $\Rightarrow$ soliton solutions

$$\zeta(x, t) = \eta_0 \operatorname{sech}^2 \left(\sqrt{\frac{3\eta_0}{h}} \frac{x - ct}{2h} \right) \quad \eta_0 = \text{constant}$$

where $c = \sqrt{\frac{gh}{1 - \eta_0/h}} \approx \sqrt{gh} \left(1 + \frac{\eta_0}{2h} \right)$

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Detailed analysis of non-linear surface waves

[Note that these derivations follow Alexander L. Fetter and John Dirk Walecka, *Theoretical Mechanics of Particles and Continua* (McGraw Hill, 1980), Chapt. 10.]

We assume that we have an incompressible fluid: $\rho = \text{constant}$

Velocity potential: $\Phi(x, z, t)$; $\mathbf{v}(x, z, t) = -\nabla \Phi(x, z, t)$

The surface of the fluid is described by $z = h + \zeta(x, t)$. It is assumed that the fluid is contained in a structure (lake, river, swimming pool, etc.) with a structureless bottom defined by the $z = 0$ plane and filled to an equilibrium height of $z = h$.

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Defining equations for $\Phi(x,z,t)$ and $\zeta(x,t)$

where $0 \leq z \leq h + \zeta(x,t)$

Continuity equation:

$$\nabla \cdot \mathbf{v} = 0 \Rightarrow \frac{\partial^2 \Phi(x,z,t)}{\partial x^2} + \frac{\partial^2 \Phi(x,z,t)}{\partial z^2} = 0$$

Bernoulli equation (assuming irrotational flow) and gravitation potential energy

$$-\frac{\partial \Phi(x,z,t)}{\partial t} + \frac{1}{2} \left[\left(\frac{\partial \Phi(x,z,t)}{\partial x} \right)^2 + \left(\frac{\partial \Phi(x,z,t)}{\partial z} \right)^2 \right] + g(z-h) = 0.$$

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Boundary conditions on functions –

Zero velocity at bottom of tank:

$$\frac{\partial \Phi(x,0,t)}{\partial z} = 0.$$

Consistent vertical velocity at water surface

$$v_z(x,z,t)|_{z=h+\zeta} = \frac{d\zeta}{dt} = \mathbf{v} \cdot \nabla \zeta + \frac{\partial \zeta}{\partial t}.$$

$$-\frac{\partial \Phi(x,z,t)}{\partial z} + \frac{\partial \Phi(x,z,t)}{\partial x} \frac{\partial \zeta(x,t)}{\partial x} - \frac{\partial \zeta(x,t)}{\partial t} \Big|_{z=h+\zeta} = 0$$

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Analysis assuming water height z is small relative to variations in the direction of wave motion (x)

Taylor's expansion about $z = 0$:

$$\Phi(x,z,t) \approx \Phi(x,0,t) + z \frac{\partial \Phi}{\partial z}(x,0,t) + \frac{z^2}{2} \frac{\partial^2 \Phi}{\partial z^2}(x,0,t) + \frac{z^3}{3!} \frac{\partial^3 \Phi}{\partial z^3}(x,0,t) + \frac{z^4}{4!} \frac{\partial^4 \Phi}{\partial z^4}(x,0,t) \dots$$

Note that the zero vertical velocity at the bottom ensures that all odd derivatives $\frac{\partial^n \Phi}{\partial z^n}(x,0,t)$ vanish from the

Taylor expansion. In addition, the Laplace equation allows us to convert all even derivatives with respect to z to derivatives with respect to x .

$$\text{Modified Taylor's expansion: } \Phi(x,z,t) \approx \Phi(x,0,t) - \frac{z^2}{2} \frac{\partial^2 \Phi}{\partial x^2}(x,0,t) + \frac{z^4}{4!} \frac{\partial^4 \Phi}{\partial x^4}(x,0,t) \dots$$

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Check linearized equations and their solutions:

Bernoulli equations --

Bernoulli equation evaluated at $z = h + \zeta(x, t)$

$$-\frac{\partial \Phi(x, h, t)}{\partial t} + g\zeta(x, t) = 0$$

Consistent vertical velocity at $z = h + \zeta(x, t)$

$$\frac{\partial \Phi(x, z, t)}{\partial z} - \frac{\partial \zeta(x, t)}{\partial t} \Big|_{z=h+\zeta} = 0$$

Using Taylor's expansion results to lowest order

$$-\frac{\partial \Phi(x, z, t)}{\partial z} \approx h \frac{\partial^2 \Phi(x, 0, t)}{\partial x^2} \quad -\frac{\partial \Phi(x, h, t)}{\partial t} \approx -\frac{\partial \Phi(x, 0, t)}{\partial t}$$

$$\text{Decoupled equations: } \frac{\partial^2 \Phi(x, 0, t)}{\partial t^2} = gh \frac{\partial^2 \Phi(x, 0, t)}{\partial x^2}.$$

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Analysis of non-linear equations -- keeping the lowest order nonlinear terms and include up to 4th order derivatives in the linear terms. Let $\phi(x, t) \equiv \Phi(x, 0, t)$

Approximate form of Bernoulli equation evaluated at surface: $z = h + \zeta$

$$-\frac{\partial \phi}{\partial t} + \frac{(h + \zeta)^2}{2} \frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \left((h + \zeta) \frac{\partial^2 \phi}{\partial x^2} \right) + g\zeta = 0$$

$$\Rightarrow -\frac{\partial \phi}{\partial t} + \frac{h^2}{2} \frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + g\zeta = 0.$$

Approximate form of surface velocity expression :

$$\frac{\partial}{\partial x} \left((h + \zeta(x, t)) \frac{\partial \phi}{\partial x} \right) - \frac{h^3}{3!} \frac{\partial^4 \phi}{\partial x^4} - \frac{\partial \zeta}{\partial t} = 0.$$

The expressions keep the lowest order nonlinear terms and include up to 4th order derivatives in the linear terms.

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Coupled equations: $-\frac{\partial \phi}{\partial t} + \frac{h^2}{2} \frac{\partial^3 \phi}{\partial t \partial x^2} + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + g\zeta = 0.$

$$\frac{\partial}{\partial x} \left((h + \zeta(x, t)) \frac{\partial \phi}{\partial x} \right) - \frac{h^3}{3!} \frac{\partial^4 \phi}{\partial x^4} - \frac{\partial \zeta}{\partial t} = 0.$$

Traveling wave solutions with new notation:

$$u \equiv x - ct \quad \phi(x, t) \equiv \chi(u) \quad \text{and} \quad \zeta(x, t) \equiv \eta(u)$$

Note that the wave "speed" c will be consistently determined

$$c \frac{d\chi(u)}{du} - \frac{ch^2}{2} \frac{d^3 \chi(u)}{du^3} + \frac{1}{2} \left(\frac{d\chi(u)}{du} \right)^2 + g\eta(u) = 0.$$

$$\frac{d}{du} \left((h + \eta(u)) \frac{d\chi(u)}{du} \right) - \frac{h^3}{6} \frac{d^4 \chi(u)}{du^4} + c \frac{d\eta(u)}{du} = 0.$$

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Integrating and re-arranging coupled equations

$$c \frac{d\chi(u)}{du} - \frac{ch^2}{2} \frac{d^3\chi(u)}{du^3} + \frac{1}{2} \left(\frac{d\chi(u)}{du} \right)^2 + g\eta(u) = 0.$$

$$\Rightarrow \chi' = -\frac{g}{c}\eta + \frac{h^2}{2}\chi''' - \frac{1}{2c}(\chi')^2 \approx -\frac{g}{c}\eta - \frac{h^2g}{2c}\eta'' - \frac{g^2}{2c^3}\eta^2$$

$$\frac{d}{du} \left((h + \eta(u)) \frac{d\chi(u)}{du} \right) - \frac{h^3}{6} \frac{d^4\chi(u)}{du^4} + c \frac{d\eta(u)}{du} = 0.$$

$$\Rightarrow (h + \eta)\chi' - \frac{h^3}{6}\chi''' + c\eta = 0$$

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Integrating and re-arranging coupled equations – continued --
Expressing modified surface velocity equation in terms of $\eta(u)$:

$$(h + \eta) \left(-\frac{g}{c}\eta - \frac{h^2g}{2c}\eta'' - \frac{g^2}{2c^3}\eta^2 \right) + \frac{h^3g}{6c}\eta'' + c\eta = 0$$

$$\Rightarrow \left(1 - \frac{gh}{c^2} \right) \eta - \frac{gh^3}{3c^2}\eta'' - \frac{g}{c^2} \left(1 + \frac{gh}{2c^2} \right) \eta^2 = 0$$

$$\Rightarrow \left(1 - \frac{hg}{c^2} \right) \eta(u) - \frac{h^2}{3}\eta''(u) - \frac{3}{2h}[\eta(u)]^2 = 0.$$

Note: $c^2 = gh + \dots$

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Solution of the famous Korteweg-de Vries equation

Modified surface amplitude equation in terms of η

$$\Rightarrow \left(1 - \frac{hg}{c^2} \right) \eta(u) - \frac{h^2}{3}\eta''(u) - \frac{3}{2h}[\eta(u)]^2 = 0.$$

Soliton solution

$$\zeta(x, t) = \eta(x - ct) = \eta_0 \operatorname{sech}^2 \left(\sqrt{\frac{3\eta_0}{h}} \frac{x - ct}{2h} \right)$$

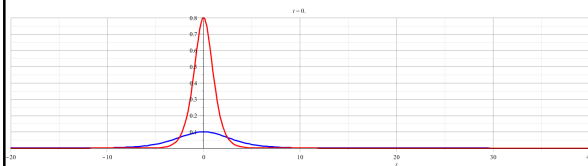
$$c = \sqrt{\frac{gh}{1 - \eta_0/h}} \approx \sqrt{gh} \left(1 + \frac{\eta_0}{2h} \right) \quad \text{where } \eta_0 \text{ is a constant}$$

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$$\zeta(x, t) = \eta(x - ct) = \eta_0 \operatorname{sech}^2 \left(\sqrt{\frac{3\eta_0}{h}} \frac{x - ct}{2h} \right)$$



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Relationship to "standard" form of Korteweg-de Vries equation

New variables:

$$\beta = 2\eta_0, \quad \bar{x} = \sqrt{\frac{3}{2h}} \frac{x}{h}, \quad \text{and} \quad \bar{t} = \sqrt{\frac{3}{2h}} \frac{ct}{2\eta_0 h}.$$

Standard Korteweg-de Vries equation

$$\frac{\partial \eta}{\partial \bar{t}} + 6\eta \frac{\partial \eta}{\partial \bar{x}} + \frac{\partial^3 \eta}{\partial \bar{x}^3} = 0.$$

Soliton solution:

$$\eta(\bar{x}, \bar{t}) = \frac{\beta}{2} \operatorname{sech}^2 \left[\frac{\sqrt{\beta}}{2} (\bar{x} - \beta \bar{t}) \right].$$

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More details

Modified surface amplitude equation in terms of η :

$$\left(1 - \frac{hg}{c^2} \right) \eta(u) - \frac{h^2}{3} \eta''(u) - \frac{3}{2h} [\eta(u)]^2 = 0.$$

Some identities: $\frac{\eta_0}{h} = 1 - \frac{gh}{c^2}$; $\frac{\partial \eta}{\partial t} = -c \frac{d\eta}{du}$; $\frac{\partial \eta}{\partial x} = \frac{d\eta}{du}$.

Derivative of surface amplitude equation:

$$\frac{\eta_0}{h} \eta' - \frac{h^2}{3} \eta''' - \frac{3}{h} \eta \eta' = 0.$$

Expression in terms of x and t :

$$-\frac{\eta_0}{ch} \frac{\partial \eta}{\partial t} - \frac{h^2}{3} \frac{\partial^3 \eta}{\partial x^3} - \frac{3}{h} \eta \frac{\partial \eta}{\partial x} = 0.$$

Expression in terms of $\bar{x}$ and $\bar{t}$:

$$\frac{\partial \eta}{\partial \bar{t}} + 6\eta \frac{\partial \eta}{\partial \bar{x}} + \frac{\partial^3 \eta}{\partial \bar{x}^3} = 0.$$

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Summary

Soliton solution

$$\zeta(x, t) = \eta(x - ct) = \eta_0 \operatorname{sech}^2 \left(\sqrt{\frac{3\eta_0}{h}} \frac{x - ct}{2h} \right)$$

$$c = \sqrt{\frac{gh}{1 - \eta_0/h}} \approx \sqrt{gh} \left(1 + \frac{\eta_0}{2h} \right) \quad \text{where } \eta_0 \text{ is a constant}$$

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Photo of canal soliton <http://www.ma.hw.ac.uk/solitons/>



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