PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

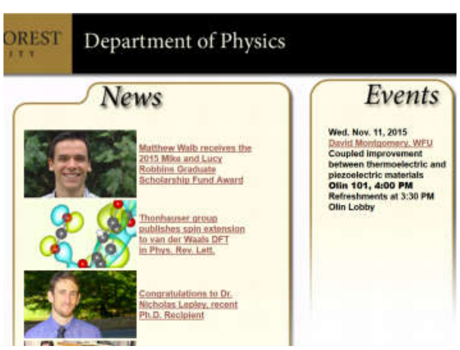
Plan for Lecture 31:

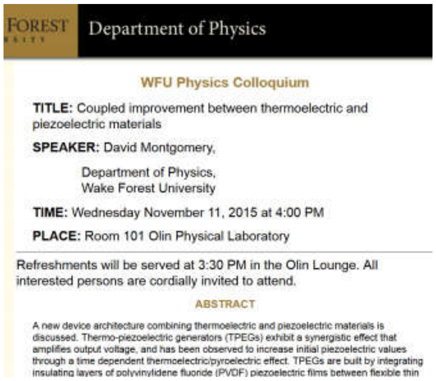
Chapter 11 in F & W:

Heat conduction

1. **Basic equations**
2. **Boundary value problems**

21	Mon, 10/19/2015	Chap. 5	Motion of Rigid Bodies	#20
22	Wed, 10/21/2015	Chap. 5	Motion of Rigid Bodies	#21
23	Fri, 10/23/2015	Chap. 8	Motion of Elastic membranes	#22
24	Mon, 10/26/2015	Chap. 9	Hydrodynamics	#23
25	Wed, 10/28/2015	Chap. 9	Hydrodynamics	#24
26	Fri, 10/30/2015	Chap. 9	Sound waves	#25
27	Mon, 11/02/2015	Chap. 9	Sound waves	#26
28	Wed, 11/04/2015	Chap. 9	Sound waves	
29	Fri, 11/06/2015	Chap. 10	Surface waves on fluids	#27
30	Mon, 11/09/2015	Chap. 10	Surface waves on fluids	#28
31	Wed, 11/11/2015	Chap. 11	Heat Conduction	#29
32	Fri, 11/13/2015	Chap. 11	Heat Conduction	
33	Mon, 11/16/2015	Chap. 11	Heat Conduction	
34	Wed, 11/18/2015	Chap. 12	Viscosity	
35	Fri, 11/20/2015	Chap. 12	More viscosity	
36	Mon, 11/23/2015	Chap. 13	Elastic Continua	
	Wed, 11/25/2015		*Thanksgiving Holiday*	
	Fri, 11/27/2015		*Thanksgiving Holiday*	
37	Mon, 11/30/2015	Chap. 13	Elastic Continua	
	Wed, 12/02/2015		Student presentations I	
	Fri, 12/04/2015		Student presentations II	
	Mon, 12/07/2015		Begin Take-home final	




FOREST Department of Physics

WFU Physics Colloquium

TITLE: Coupled improvement between thermoelectric and piezoelectric materials

SPEAKER: David Montgomery,

Department of Physics,
Wake Forest University

TIME: Wednesday November 11, 2015 at 4:00 PM

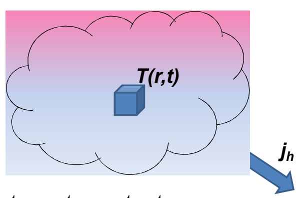
PLACE: Room 101 Olin Physical Laboratory

Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

ABSTRACT

A new device architecture combining thermoelectric and piezoelectric materials is discussed. Thermo-piezoelectric generators (TPEGs) exhibit a synergistic effect that amplifies output voltage, and has been observed to increase initial piezoelectric values through a time dependent thermoelectric/pyroelectric effect. TPEGs are built by integrating insulating layers of polyvinylidene fluoride (PVDF) piezoelectric films between flexible thin

Conduction of heat



Enthalpy of a system at constant pressure p

non uniform temperature $T(\mathbf{r},t)$

mass density ρ and heat capacity c_p

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0\right) d^3r + H_0(T_0, p)$$

Note that in this treatment we are considering a system at constant pressure *p*

Notation:	Heat added to system	-- $dQ = TdS$
	External work done on system	-- $dW = -pdV$
	Internal energy	-- $dE = dQ + dW = TdS - pdV$
	Entropy	-- dS
	Enthalpy	-- $dH = d(E + pV) = TdS + Vdp$

Heat capacity at constant pressure:

$$C_p \equiv \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial H}{\partial T}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p$$

$$C_p = \rho c_p d^3 r$$

Conduction of heat -- continued

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0\right) d^3r + H_0(T_0, p)$$

Time rate of change of enthapy:

$$\frac{dH}{dt} = \int_V \rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} d^3r = -\int_A \mathbf{j}_h \cdot d\mathbf{A} + \int_V \rho \dot{q} d^3r$$

heat flux

heat source

$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

Conduction of heat -- continued

$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \cdot \mathbf{j}_h + \rho \dot{q}$$

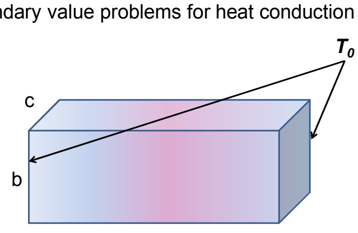
Empirically: $\mathbf{j}_h = -k_{th} \nabla T(\mathbf{r},t)$

$$\Rightarrow \frac{\partial T(\mathbf{r},t)}{\partial t} = \kappa \nabla^2 T(\mathbf{r},t) + \frac{\dot{q}}{c_p}$$

$\kappa \equiv \frac{k_{th}}{\rho c_p}$ thermal diffusivity

	Typical values (m^2/s)
Air	2×10^{-5}
Water	1×10^{-7}
Copper	1×10^{-4}

Boundary value problems for heat conduction



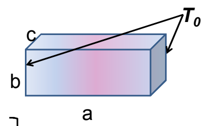
$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = \frac{\dot{q}}{c_p}$$

Without source term: $\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = 0$

Example with boundary values: $T(0,y,z,t) = T(a,y,z,t) = T_0$

Boundary value problems for heat conduction

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$



$$T(0,y,z,t) = T(a,y,z,t) = T_0$$

$$\frac{\partial T(x,0,z,t)}{\partial y} = \frac{\partial T(x,b,z,t)}{\partial y} = 0$$

$$\frac{\partial T(x,y,0,t)}{\partial z} = \frac{\partial T(x,y,c,t)}{\partial z} = 0$$

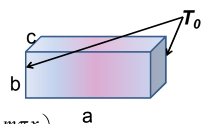
Assuming thermally insulated boundaries

Separation of varables : $T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$

Let $\frac{d^2X}{dx^2} = -\alpha^2 X \quad \frac{d^2Y}{dy^2} = -\beta^2 Y \quad \frac{d^2Z}{dz^2} = -\gamma^2 Z$

$$\Rightarrow -\lambda + \kappa(\alpha^2 + \beta^2 + \gamma^2) = 0$$

Boundary value problems for heat conduction



$$T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$$

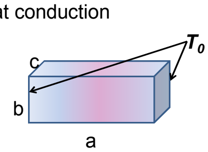
$$X(0) = X(a) = 0 \qquad \Rightarrow X(x) = \sin\left(\frac{m\pi x}{a}\right)$$

$$\frac{dY(0)}{dy} = \frac{dY(b)}{dy} = 0 \qquad \Rightarrow Y(y) = \cos\left(\frac{n\pi y}{b}\right)$$

$$\frac{dZ(0)}{dz} = \frac{dZ(c)}{dz} = 0 \qquad \Rightarrow Z(z) = \cos\left(\frac{p\pi z}{c}\right)$$

$$-\lambda_{nmp} + \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right) = 0$$

Boundary value problems for heat conduction



Full solution:

$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp} t}$$

$$\lambda_{nmp} = \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right)$$

Initial value problem in an infinite domain; Fourier transform

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

$$T(\mathbf{r},0) = f(\mathbf{r})$$

$$\text{Let : } \quad \widetilde{T}(\mathbf{q},t) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} T(\mathbf{r},t)$$

$$\widetilde{f}(\mathbf{q}) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} f(\mathbf{r})$$

$$\Rightarrow \widetilde{T}(\mathbf{q},0) = \widetilde{f}(\mathbf{q})$$

$$\Rightarrow \frac{\partial \widetilde{T}(\mathbf{q},t)}{\partial t} = -\kappa q^2 \widetilde{T}(\mathbf{q},t)$$

$$\widetilde{T}(\mathbf{q},t) = \widetilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

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Initial value problem in an infinite domain; Fourier transform

$$\widetilde{T}(\mathbf{q},t) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} T(\mathbf{r},t) \qquad \Rightarrow T(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{r}} \widetilde{T}(\mathbf{q},t)$$

$$\widetilde{T}(\mathbf{q},t) = \widetilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

$$T(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{r}} \widetilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

$$\widetilde{T}(\mathbf{q},0) = \widetilde{f}(\mathbf{q}) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} f(\mathbf{r})$$

$$T(\mathbf{r},t) = \int d^3r' G(\mathbf{r}-\mathbf{r}',t) T(\mathbf{r}',0)$$

$$\text{with } \; G(\mathbf{r}-\mathbf{r}',t) \equiv \frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} e^{-\kappa q^2 t}$$

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Initial value problem in an infinite domain; Fourier transform

$$T(\mathbf{r},t) = \int d^3r' G(\mathbf{r}-\mathbf{r}',t) T(\mathbf{r}',0)$$

$$\text{with } \; G(\mathbf{r}-\mathbf{r}',t) \equiv \frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} e^{-\kappa q^2 t}$$

$$G(\mathbf{r}-\mathbf{r}',t) = \frac{1}{(4\pi\kappa t)^{3/2}} e^{-|\mathbf{r}-\mathbf{r}'|^2/(4\kappa t)}$$

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Heat equation in half-space

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = 0$$

$T(\mathbf{r},t) \Rightarrow T(z,t)$ with initial and boundary values :

$T(z,t) \equiv 0$ for $z < 0$

$T(z,0) = 0$ for $z > 0$

$T(0,t) = T_0$ for $t \geq 0$

Solution : $T = T_0 \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$

where $\operatorname{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$

Heat equation in half-space -- continued

$$\frac{\partial T(z,t)}{\partial t} - \kappa \frac{\partial^2 T(z,t)}{\partial z^2} = 0$$

Solution : $T = T_0 \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$

where $\operatorname{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$

Note that $\frac{d \operatorname{erfc}(x)}{dx} = \frac{d}{dx} \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du = -\frac{2}{\sqrt{\pi}} e^{-x^2}$

$$\frac{\partial}{\partial t} \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)} \left(\frac{z}{4\sqrt{\kappa t^3}}\right)$$

$$\frac{\partial^2}{\partial z^2} \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)} \left(\frac{z}{4\kappa\sqrt{\kappa t^3}}\right)$$

$$T = T_0 \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$