PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 32

Viscous fluids – Chap. 12 in F & W

1. **Viscous stress tensor**
2. **Navier-Stokes equation**
3. **Example for incompressible fluid**

22	Wed, 10/21/2015	Chap. 5	Motion of Rigid Bodies	#21
23	Fri, 10/23/2015	Chap. 8	Motion of Elastic membranes	#22
24	Mon, 10/26/2015	Chap. 9	Hydrodynamics	#23
25	Wed, 10/28/2015	Chap. 9	Hydrodynamics	#24
26	Fri, 10/30/2015	Chap. 9	Sound waves	#25
27	Mon, 11/02/2015	Chap. 9	Sound waves	#26
28	Wed, 11/04/2015	Chap. 9	Sound waves	
29	Fri, 11/06/2015	Chap. 10	Surface waves on fluids	#27
30	Mon, 11/09/2015	Chap. 10	Surface waves on fluids	#28
31	Wed, 11/11/2015	Chap. 11	Heat Conduction	#29
32	Fri, 11/13/2015	Chap. 12	Viscosity	#30
33	Mon, 11/16/2015	Chap. 12	Viscosity	
34	Wed, 11/18/2015	Chap. 12	Viscosity	
35	Fri, 11/20/2015	Chap. 12	More viscosity	
36	Mon, 11/23/2015	Chap. 13	Elastic Continua	
	Wed, 11/25/2015		*Thanksgiving Holiday*	
	Fri, 11/27/2015		*Thanksgiving Holiday*	
37	Mon, 11/30/2015	Chap. 13	Elastic Continua	
	Wed, 12/02/2015		Student presentations I	
	Fri, 12/04/2015		Student presentations II	
	Mon, 12/07/2015		Begin Take-home final	

General formulation of viscosity from Chapter 12 of Fetter and Walecka

For a non-viscous fluid, we can define a stress tensor:

$$T_{kl}^{ideal} = \rho v_k v_l + p\delta_{kl}$$

From Euler and Newton equations for fluid:

$$\int_V d^3r \frac{\partial(\rho v_k)}{\partial t} = -\sum_{l=1}^{3}\int_A dA_l T_{kl} + \int_V d^3r \rho f_k$$

$$-\sum_{l=1}^{3}\int_A dA_l T_{kl} = k\text{th component of force acting on surface } dA$$

For an ideal (non-viscous) fluid: $T_{kl} = T_{lk}$

Differential form of momentum conservation law:

$$\frac{\partial(\rho v_i)}{\partial t} + \sum_{i=1}^{3}\frac{\partial T_{ij}}{\partial x_j} = \rho f_i$$

Note that in absence of viscous effect, the stress tensor relations are identical to the Newton-Euler and continuity equations:

Differential form of momentum conservation law in terms of stress tensor:

$$\frac{\partial(\rho v_i)}{\partial t} + \sum_{i=1}^{3} \frac{\partial T_{ij}}{\partial x_j} = \rho f_i$$

Newton-Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Effects of viscosity

Argue that viscosity is due to shear forces in a fluid of the form:

$$F_{drag} = \eta \frac{\partial v_x}{\partial y} dA$$

Formulate viscosity stress tensor with traceless and diagonal terms:

$$T^v_{kl} = -\eta\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3}\delta_{kl}(\nabla \cdot \mathbf{v})\right) - \zeta\delta_{kl}(\nabla \cdot \mathbf{v})$$

viscosity

bulk viscosity

Total stress tensor: $T_{kl} = T^{ideal}_{kl} + T^v_{kl}$

$$T^{ideal}_{kl} = \rho v_k v_l + p\delta_{kl}$$

$$T^v_{kl} = -\eta\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} - \frac{2}{3}\delta_{kl}(\nabla \cdot \mathbf{v})\right) - \zeta\delta_{kl}(\nabla \cdot \mathbf{v})$$

Effects of viscosity -- continued

Incorporating generalized stress tensor into Newton-Euler equations

$$\frac{\partial \rho v_i}{\partial t} + \sum_{j=1}^{3} \frac{\partial \rho v_i v_j}{\partial x_j} = \rho f_i - \frac{\partial p}{\partial x_i} + \eta \sum_{j=1}^{3} \frac{\partial^2 v_i}{\partial x_j^2} + \left(\zeta + \frac{1}{3}\eta\right) \sum_{j=1}^{3} \frac{\partial^2 v_j}{\partial x_i \partial x_j}$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^{3} \frac{\partial \rho v_j}{\partial x_j} = 0$$

Vector form (Navier-Stokes equation)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \frac{1}{\rho}\left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla \cdot \mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Newton-Euler equations for viscous fluids

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}-\frac{1}{\rho}\nabla p+\frac{\eta}{\rho}\nabla^2\mathbf{v}+\frac{1}{\rho}\left(\zeta+\frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0$$

Typical viscosities at 20^o C and 1 atm:

Fluid	η/ρ (m^2/s)	η (Pa s)
Water	1.00×10^{-6}	1×10^{-3}
Air	14.9×10^{-6}	0.018×10^{-3}
Ethyl alcohol	1.52×10^{-6}	1.2×10^{-3}
Glycerine	1183×10^{-6}	1490×10^{-3}

Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R*

Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\mathbf{f}-\frac{1}{\rho}\nabla p+\frac{\eta}{\rho}\nabla^2\mathbf{v}+\frac{1}{\rho}\left(\zeta+\frac{1}{3}\eta\right)\nabla(\nabla\cdot\mathbf{v})$$

Continuity condition

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0$$

Incompressible fluid $\Rightarrow \nabla\cdot\mathbf{v}=0$

Steady flow $\Rightarrow \frac{\partial \mathbf{v}}{\partial t}=0$

Irrotational flow $\Rightarrow \nabla\times\mathbf{v}=0$

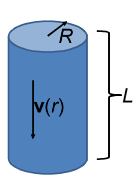
No applied force $\Rightarrow \mathbf{f}=0$

Neglect non-linear terms $\Rightarrow \nabla\left(v^2\right)=0$

Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R* -- continued

Navier-Stokes equation becomes:

$$0=-\frac{1}{\rho}\nabla p+\frac{\eta}{\rho}\nabla^2\mathbf{v}$$



Assume that $\mathbf{v}(\mathbf{r},t)=v_z(r)\hat{\mathbf{z}}$

$$\frac{\partial p}{\partial z}=\eta\nabla^2 v_z(r) \quad \text{(independent of } z\text{)}$$

Suppose that $\frac{\partial p}{\partial z}=-\frac{\Delta p}{L}$ (uniform pressure gradient)

$$\Rightarrow \nabla^2 v_z(r)=-\frac{\Delta p}{\eta L}$$

Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R* -- continued

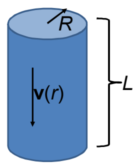
$$\nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

$$\frac{1}{r}\frac{d}{dr} r \frac{dv_z(r)}{dr} = -\frac{\Delta p}{\eta L}$$

$$v_z(r) = -\frac{\Delta p r^2}{4\eta L} + C_1 \ln(r) + C_2$$

$$\Rightarrow C_1 = 0 \qquad v_z(R) = 0 = -\frac{\Delta p R^2}{4\eta L} + C_2$$

$$v_z(r) = \frac{\Delta p}{4\eta L}\left(R^2 - r^2\right)$$

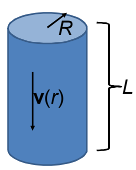


Example – steady flow of an incompressible fluid in a long pipe with a circular cross section of radius *R* -- continued

$$v_z(r) = \frac{\Delta p}{4\eta L}\left(R^2 - r^2\right)$$

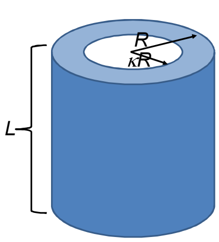
Mass flow rate through the pipe:

$$\frac{dM}{dt} = 2\pi\rho \int_0^R r\,dr\,v_z(r) = \frac{\Delta p \rho \pi R^4}{8\eta L}$$



Poiseuille formula;
➔Method for measuring η

Example – steady flow of an incompressible fluid in a long tube with a circular cross section of outer radius *R* and inner radius *κR*



$$\nabla^2 v_z(r) = -\frac{\Delta p}{\eta L}$$

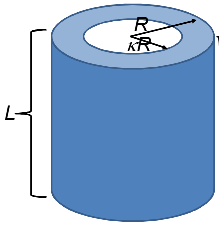
$$\frac{1}{r}\frac{d}{dr} r \frac{dv_z(r)}{dr} = -\frac{\Delta p}{\eta L}$$

$$v_z(r) = -\frac{\Delta p r^2}{4\eta L} + C_1 \ln(r) + C_2$$

$$v_z(R) = 0 = -\frac{\Delta p R^2}{4\eta L} + C_1 \ln(R) + C_2$$

$$v_z(\kappa R) = 0 = -\frac{\Delta p \kappa^2 R^2}{4\eta L} + C_1 \ln(\kappa R) + C_2$$

Example – steady flow of an incompressible fluid in a long tube with a circular cross section of outer radius *R* and inner radius κR -- continued



Solving for C_1 and C_2 :

$$v_z(r) = -\frac{\Delta p R^2}{4\eta L}\left(1-\left(\frac{r}{R}\right)^2+\frac{1-\kappa^2}{\ln\kappa}\ln\left(\frac{R}{r}\right)\right)$$

Mass flow rate through the pipe:

$$\frac{dM}{dt} = 2\pi\rho\int_{\kappa R}^{R} r dr v_z(r) = \frac{\Delta p \rho \pi R^4}{8\eta L}\left(1-\kappa^4+\frac{\left(1-\kappa^2\right)^2}{\ln\kappa}\right)$$