PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 33:
Effects of viscosity in fluid motion – Chap.12 in Fetter & Walecka

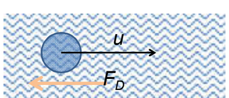
1. **Navier-Stokes equation**
2. **Terminal velocity of a sphere moving with constant applied force in a viscous medium**
3. **Stokes' viscosity relation**

22	Wed, 10/21/2015	Chap. 5	Motion of Rigid Bodies	#21
23	Fri, 10/23/2015	Chap. 8	Motion of Elastic membranes	#22
24	Mon, 10/26/2015	Chap. 9	Hydrodynamics	#23
25	Wed, 10/28/2015	Chap. 9	Hydrodynamics	#24
26	Fri, 10/30/2015	Chap. 9	Sound waves	#25
27	Mon, 11/02/2015	Chap. 9	Sound waves	#26
28	Wed, 11/04/2015	Chap. 9	Sound waves	
29	Fri, 11/06/2015	Chap. 10	Surface waves on fluids	#27
30	Mon, 11/09/2015	Chap. 10	Surface waves on fluids	#28
31	Wed, 11/11/2015	Chap. 11	Heat Conduction	#29
32	Fri, 11/13/2015	Chap. 12	Viscosity	#30
33	Mon, 11/16/2015	Chap. 12	Viscosity	Prepare presentation.
34	Wed, 11/18/2015	Chap. 12	Viscosity	Prepare presentation.
35	Fri, 11/20/2015	Chap. 12	More viscosity	Prepare presentation.
36	Mon, 11/23/2015	Chap. 13	Elastic Continua	Prepare presentation.
	Wed, 11/25/2015		*Thanksgiving Holiday*	
	Fri, 11/27/2015		*Thanksgiving Holiday*	
37	Mon, 11/30/2015	Chap. 13	Elastic Continua	Prepare presentation.
	Wed, 12/02/2015		Student presentations I	
	Fri, 12/04/2015		Student presentations II	
	Mon, 12/07/2015		Begin Take-home final	

Brief introduction to viscous effects in incompressible fluids

Stokes' analysis of viscous drag on a sphere of radius R moving at speed u in medium with viscosity η:

$$F_D = -\eta(6\pi R u)$$



Plan:
1. Consider the general effects of viscosity on fluid equations
2. Consider the solution to the linearized equations for the case of steady-state flow of a sphere of radius R
3. Infer the drag force needed to maintain the steady-state flow

Newton-Euler equation for incompressible fluid, modified by viscous contribution (Navier-Stokes equation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} + \underbrace{\frac{\eta}{\rho}}_{\upsilon}\nabla^2\mathbf{v}$$

υ Kinematic viscosity

Typical kinematic viscosities at 20° C and 1 atm:

Fluid	ν (m²/s)
Water	1.00 x 10^{-6}
Air	14.9 x 10^{-6}
Ethyl alcohol	1.52 x 10^{-6}
Glycerine	1183 x 10^{-6}

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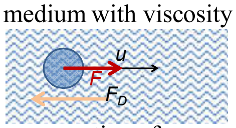

Effects of drag force on motion of particle of mass m with constant force F:

$$F - 6\pi R\eta u = m\frac{du}{dt} \qquad \text{with } u(0) = 0$$

$$\Rightarrow u(t) = \frac{F}{6\pi R\eta}\left(1 - e^{-\frac{6\pi R\eta}{m}t}\right)$$

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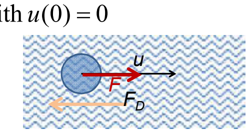
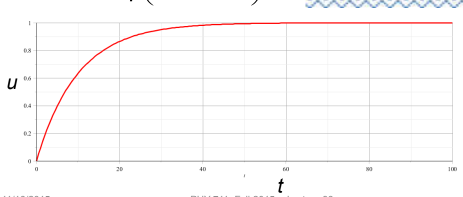
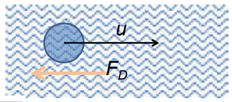


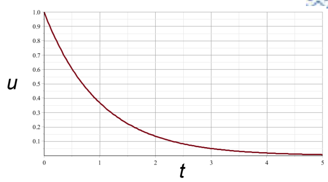


Effects of drag force on motion of particle of mass m with an initial velocity with $u(0) = U_0$ and no external force

$$-6\pi R\eta u = m\frac{du}{dt}$$

$$\Rightarrow u(t) = U_0 e^{-\frac{6\pi R\eta}{m}t}$$



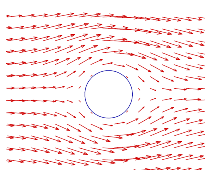


Recall: PHY 711 -- Assignment #24 Oct. 28, 2015

Determine the form of the velocity potential for an incompressible fluid representing uniform velocity in the **z** direction at large distances from a spherical obstruction of radius *a*. Find the form of the velocity potential and the velocity field for all *r* > *a*. Assume that the velocity in the radial direction is 0 for *r* = *a* and assume that the velocity is uniform in the azimuthal direction.

$$\nabla^2\Phi = 0$$

$$\Phi(r,\theta) = \left(-v_0 r + \frac{v_0 R^3}{2r^2}\right)\cos\theta$$



Newton-Euler equation for incompressible fluid, modified by viscous contribution (Navier-Stokes equation):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} + \frac{\eta}{\rho}\nabla^2\mathbf{v}$$

Continuity equation: $\nabla \cdot \mathbf{v} = 0$

Assume steady state: $\Rightarrow \frac{\partial \mathbf{v}}{\partial t} = 0$

Assume non-linear effects small

Initially set $\mathbf{f}_{applied} = 0;$

$$\Rightarrow \nabla p = \eta \nabla^2 \mathbf{v}$$

$$\nabla p = \eta \nabla^2 \mathbf{v}$$

Take curl of both sides of equation:

$$\nabla \times (\nabla p) = 0 = \eta \nabla^2 (\nabla \times \mathbf{v})$$

Assume (with a little insight from Landau):

$$\mathbf{v} = \nabla \times (\nabla \times f(r)\mathbf{u}) + \mathbf{u}$$

where $f(r) \xrightarrow[r\to\infty]{} 0$

Note that:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Digression

Some comment on assumption: $\mathbf{v} = \nabla \times (\nabla \times f(r)\mathbf{u}) + \mathbf{u}$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Here $\mathbf{A} = \nabla \times f(r)\mathbf{u}$

$$\nabla \times \mathbf{v} = 0 = \nabla \times (\nabla \times (\nabla \times \mathbf{A})) = -\nabla \times (\nabla^2 \mathbf{A})$$

Also note : $\nabla p = \eta \nabla^2 \mathbf{v}$

$\Rightarrow \nabla \times \nabla p = \nabla \times \eta \nabla^2 \mathbf{v}$ or $\nabla^2 (\nabla \times \mathbf{v}) = 0$

$$\mathbf{v} = \nabla \times (\nabla \times f(r)\mathbf{u}) + \mathbf{u}$$

$$\mathbf{u} = u\hat{\mathbf{z}}$$

$$\nabla \times (\nabla \times f(r)\hat{\mathbf{z}}) = \nabla(\nabla \cdot f(r)\hat{\mathbf{z}}) - \nabla^2 f(r)\hat{\mathbf{z}}$$

$$\nabla \times \mathbf{v} = 0 \qquad \Rightarrow \nabla^2 (\nabla \times \mathbf{v}) = 0$$

$$\nabla^4 (\nabla \times f(r)\hat{\mathbf{z}}) = 0 \quad \Rightarrow \nabla^4 (\nabla f(r) \times \hat{\mathbf{z}}) = 0 \quad \Rightarrow \nabla^4 f(r) = 0$$

$$f(r) = C_1 r^2 + C_2 r + C_3 + \frac{C_4}{r}$$

$$v_r = u\cos\theta\left(1 - \frac{2}{r}\frac{df}{dr}\right) = u\cos\theta\left(1 - 4C_1 - \frac{2C_2}{r} - \frac{2C_4}{r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1 - \frac{d^2 f}{dr^2} - \frac{1}{r}\frac{df}{dr}\right) = -u\sin\theta\left(1 - 4C_1 - \frac{C_2}{r} + \frac{C_4}{r^3}\right)$$

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To satisfy $\mathbf{v}(r \to \infty) = \mathbf{u}$: $\Rightarrow C_1 = 0$

To satisfy $\mathbf{v}(R) = 0$ solve for C_2, C_4

$$v_r = u\cos\theta\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)$$

$$v_\theta = -u\sin\theta\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)$$

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$$v_r = u\cos\theta\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)$$

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Determining pressure:

$$\nabla p = \eta\nabla^2\mathbf{v} = -\eta\nabla\left(u\cos\theta\left(\frac{3R}{2r^2}\right)\right)$$

$$\Rightarrow p = p_0 - \eta u\cos\theta\left(\frac{3R}{2r^2}\right)$$

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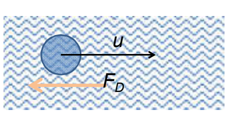
$$p = p_0 - \eta u\cos\theta\left(\frac{3R}{2r^2}\right)$$

Corresponds to:

$$F_D\cos\theta = \left(p(R) - p_0\right)4\pi R^2$$

$$\Rightarrow \mathrm{F}_D = -\eta u\left(6\pi R\right)$$



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