PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

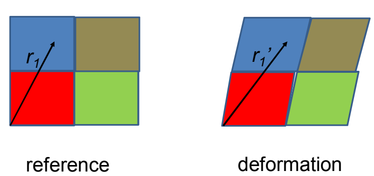
Plan for Lecture 35

Physics of elastic continua – Chap. 13 in F & W

1. Stress and strain
2. Waves in elastic media

23	Fri, 10/23/2015	Chap. 8	Motion of Elastic membranes	#22
24	Mon, 10/26/2015	Chap. 9	Hydrodynamics	#23
25	Wed, 10/28/2015	Chap. 9	Hydrodynamics	#24
26	Fri, 10/30/2015	Chap. 9	Sound waves	#25
27	Mon, 11/02/2015	Chap. 9	Sound waves	#26
28	Wed, 11/04/2015	Chap. 9	Sound waves	
29	Fri, 11/06/2015	Chap. 10	Surface waves on fluids	#27
30	Mon, 11/09/2015	Chap. 10	Surface waves on fluids	#28
31	Wed, 11/11/2015	Chap. 11	Heat Conduction	#29
32	Fri, 11/13/2015	Chap. 12	Viscosity	#30
33	Mon, 11/16/2015	Chap. 12	Viscosity	Prepare presentation.
34	Wed, 11/18/2015	Chap. 12	Viscosity	Prepare presentation.
35	Fri, 11/20/2015	Chap. 12	Elastic Continua	Prepare presentation.
36	Mon, 11/23/2015	Chap. 13	Review of Mathematical Methods	Prepare presentation.
	Wed, 11/25/2015		*Thanksgiving Holiday*	
	Fri, 11/27/2015		*Thanksgiving Holiday*	
37	Mon, 11/30/2015	Chap. 13	Review of Mathematical Methods	Prepare presentation.
	Wed, 12/02/2015		Student presentations I	
	Fri, 12/04/2015		Student presentations II	
	Mon, 12/07/2015		Begin Take-home final	

Brief introduction to elastic continua



$$\mathbf{r}_1' = \mathbf{r}_1 + \mathbf{u}(\mathbf{r}_1) \qquad \mathbf{r}_2' = \mathbf{r}_2 + \mathbf{u}(\mathbf{r}_2)$$

$$\mathbf{r}_2' - \mathbf{r}_1' = \mathbf{r}_2 - \mathbf{r}_1 + \left((\mathbf{r}_2 - \mathbf{r}_1)\cdot\nabla\right)\mathbf{u}(\mathbf{r}_1)$$

Brief introduction to elastic continua

Deformation components:

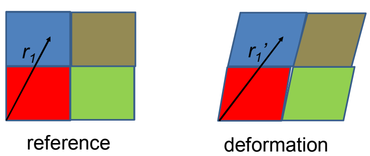
$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\equiv \epsilon_{ij} + O_{ij}$$

elastic strain tensor (ϵ_{ij})

rotation of material (O_{ij})

Brief introduction to elastic continua



$$\mathbf{r}_1' = \mathbf{r}_1 + \mathbf{u}(\mathbf{r}_1) \qquad \mathbf{r}_2' = \mathbf{r}_2 + \mathbf{u}(\mathbf{r}_2)$$

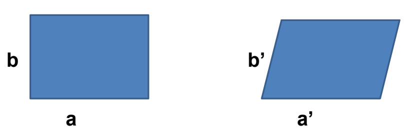
$$\mathbf{r}_2' - \mathbf{r}_1' = \mathbf{r}_2 - \mathbf{r}_1 + \left((\mathbf{r}_2 - \mathbf{r}_1)\cdot\nabla\right)\mathbf{u}(\mathbf{r}_1)$$

Effects of strain on a vector:

$$\mathbf{a}' = \mathbf{a} + a\left(\epsilon_{11}\hat{\mathbf{x}} + \epsilon_{21}\hat{\mathbf{y}} + \epsilon_{31}\hat{\mathbf{z}}\right)$$

$$a' = |\mathbf{a}'\cdot\mathbf{a}'| \approx a\left(1+\epsilon_{11}\right)$$

Deformation



$$\mathbf{a}' = \mathbf{a} + a\left(\epsilon_{11}\hat{\mathbf{x}} + \epsilon_{21}\hat{\mathbf{y}} + \epsilon_{31}\hat{\mathbf{z}}\right)$$

$$\mathbf{b}' = \mathbf{b} + b\left(\epsilon_{12}\hat{\mathbf{x}} + \epsilon_{22}\hat{\mathbf{y}} + \epsilon_{32}\hat{\mathbf{z}}\right)$$

for $\mathbf{a}\cdot\mathbf{b} = 0$

$$\mathbf{a}'\cdot\mathbf{b}' \approx ab\left(\epsilon_{21} + \epsilon_{12}\right)$$

$$\theta' \approx \theta - 2\epsilon_{12}$$

Brief introduction to elastic continua

Deformation components:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$\equiv \epsilon_{ij} + O_{ij}$$

$$V = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \qquad V' = \mathbf{a}'\cdot(\mathbf{b}'\times\mathbf{c}') \qquad V' = V\left(1+\mathrm{Tr}(\epsilon)\right) = V(1+\nabla\cdot\mathbf{u})$$

$$\mathrm{Tr}(\epsilon) = \nabla\cdot\mathbf{u} = \frac{dV}{V} = -\frac{d\rho}{\rho}$$

Stress tensor

$-\sum_{j=1}^{3} T_{ij} dA_j \Rightarrow i^{th}$ component of force acting on surface $\hat{\mathbf{n}}$

Generalization of Hooke's law, $F_x = -kx$:

Lame' coefficients : $T_{ij} = -\lambda\delta_{ij}\nabla\cdot\mathbf{u} - \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

or : $T_{ij} = -\lambda\delta_{ij}\mathrm{Tr}(\epsilon) - 2\mu\epsilon_{ij}$

Note that: $\mathrm{Tr}(T) = -3\underbrace{\left(\lambda + \frac{2}{3}\mu\right)}_{K}\mathrm{Tr}(\epsilon)$

$K \equiv$ bulk modulus$= -V\left(\frac{\partial p}{\partial V}\right)$

$$\Rightarrow \epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)}\delta_{ij} Tr(T)\right)$$

Stress tensor -- continued

In terms of bulk modulus: $K = \lambda + \frac{2}{3}\mu$

$$\epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)}\delta_{ij} Tr(T)\right)$$

$$\epsilon_{ij} = -\frac{1}{9K}\delta_{ij} Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\delta_{ij} Tr(T)\right)$$

$$\epsilon_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\left(\lambda + \frac{2}{3}\mu\right)}\,\delta_{ij}Tr(T)\right) = -\frac{1}{9K}\delta_{ij}Tr(T) - \frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}\,\delta_{ij}Tr(T)\right)$$

Example -- hydrostatic pressure: $T_{ij} = \delta_{ij}dp$

$$\epsilon_{ij} = -\frac{dp}{3\left(\lambda + \frac{2}{3}\mu\right)}\delta_{ij} \equiv -\frac{dp}{3K}\delta_{ij}$$

$$\Rightarrow K = -V\frac{\partial p}{\partial V}$$

Example -- uniaxial pressure: $T_{ij} = \begin{cases} dp & ij = zz \\ 0 & \text{otherwise} \end{cases}$

$$\epsilon_{zz} = -\frac{1}{E}T_{zz} \qquad \text{in terms of Young's modulus}$$

$$E = \frac{9K\mu}{3K + \mu}$$

Example -- uniaxial pressure: $T_{ij} = \begin{cases} dp & ij = zz \\ 0 & \text{otherwise} \end{cases}$

transverse contributions:

$$\epsilon_{xx} = \epsilon_{yy} = \left(-\frac{1}{9K} + \frac{1}{6\mu}\right)T_{zz}$$

Poisson's ratio:

$$\sigma = -\frac{\epsilon_{xx}}{\epsilon_{zz}} = \frac{1}{2}\frac{3K - 2\mu}{3K + \mu}$$

Relationships between elastic constants:

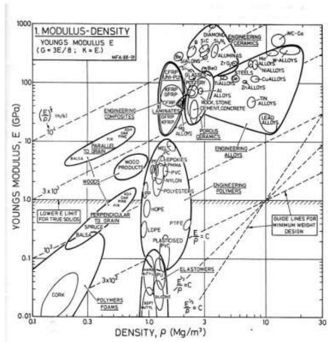
$$K = \frac{1}{3}\frac{E}{1 - 2\sigma}$$

$$\mu = \frac{1}{2}\frac{E}{1 + \sigma}$$

Shear modulus

$$T_{ij} = \begin{cases} -f & \text{for } T_{xy} \text{ or } T_{yx} \\ 0 & \text{otherwise} \end{cases}$$

$$\epsilon_{xy} = \epsilon_{yx} = \frac{f}{\mu}$$



From: Material Selection in Mechanical Design, M.F Ashby, Pergamon Press, Oxford, 1992

Dynamical equations of elastic continuum

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu \nabla^2 \mathbf{u} + \left(K + \frac{1}{3}\mu \right) \nabla \left(\nabla \cdot \mathbf{u} \right) + \rho \mathbf{f}$$

In the absence of external forces, this reduces to two decoupled wave equations representing longitudinal and transverse modes:

$\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$

where $\nabla \times \mathbf{u}_l = 0$ and $\nabla \cdot \mathbf{u}_t = 0$

$$c_l = \left(\frac{K + \frac{4}{3}\mu}{\rho} \right)^{1/2} \quad \text{and} \quad c_t = \left(\frac{\mu}{\rho} \right)^{1/2}$$

from:
https://pangea.stanford.edu/courses/gp262/Notes/5.Elasticity.pdf

Mineral	Density	Young's Modulus	Bulk Modulus	Shear Modulus	Vp	Vs	Poisson's Ratio
Quartz	2.6500	95.756	36.600	45.000	6.0376	4.1208	0.063953
Calcite	2.7100	84.293	76.800	32.000	6.6395	3.4363	0.31707
Dolomite	2.8700	116.57	94.900	45.000	7.3465	3.9597	0.29527
Clay (kaolinite)	1.5800	3.2034	1.5000	1.4000	1.4597	0.94132	0.14407
Muscovite	2.7900	100.84	61.500	41.100	6.4563	3.8381	0.22673
Feldspar (Albite)	2.6300	69.010	75.600	25.600	6.4594	3.1199	0.34786
Halite	2.1600	37.242	24.800	14.900	4.5474	2.6264	0.24972
Anhydrite	2.9800	74.431	56.100	29.100	5.6432	3.1249	0.27888
Pyrite	4.9300	305.85	147.40	132.50	8.1076	5.1842	0.15417
Siderite	3.9600	134.51	123.70	51.000	6.9576	3.5887	0.31876
gas	0.00065000	0.0000	0.00013000	0.0000	0.44721	0.0000	0.50000
water	1.0000	0.0000	2.2500	0.0000	1.5000	0.0000	0.50000
oil	0.80000	0.0000	1.0200	0.0000	1.1292	0.0000	0.50000

C.1

densities in g/cm^3
moduli in GPa
velocities in km/s