PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 36

Review of Mathematical Methods

1. More details on Green's function methods
2. Special functions

23	Fri, 10/23/2015	Chap. 8	Motion of Elastic membranes	#22
24	Mon, 10/26/2015	Chap. 9	Hydrodynamics	#23
25	Wed, 10/28/2015	Chap. 9	Hydrodynamics	#24
26	Fri, 10/30/2015	Chap. 9	Sound waves	#25
27	Mon, 11/02/2015	Chap. 9	Sound waves	#26
28	Wed, 11/04/2015	Chap. 9	Sound waves	
29	Fri, 11/06/2015	Chap. 10	Surface waves on fluids	#27
30	Mon, 11/09/2015	Chap. 10	Surface waves on fluids	#28
31	Wed, 11/11/2015	Chap. 11	Heat Conduction	#29
32	Fri, 11/13/2015	Chap. 12	Viscosity	#30
33	Mon, 11/16/2015	Chap. 12	Viscosity	Prepare presentation.
34	Wed, 11/18/2015	Chap. 12	Viscosity	Prepare presentation.
35	Fri, 11/20/2015	Chap. 12	Elastic Continua	Prepare presentation.
36	Mon, 11/23/2015	Chap. 13	Review of Mathematical Methods	Prepare presentation.
	Wed, 11/25/2015		*Thanksgiving Holiday*	
	Fri, 11/27/2015		*Thanksgiving Holiday*	
37	Mon, 11/30/2015	Chap. 13	Review of Mathematical Methods	Prepare presentation.
	Wed, 12/02/2015		Student presentations I	
	Fri, 12/04/2015		Student presentations II	
	Mon, 12/07/2015		Begin Take-home final	

Linear second-order ordinary differential equations
Sturm-Liouville equations

Inhomogenous problem: $$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

given functions

applied force

solution to be determined

Solution to inhomogeneous problem by using Green's functions

Inhomogenous problem:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Formal solution:

$$\varphi_\lambda(x)=\varphi_{\lambda 0}(x)+\int G_\lambda(x,x')F(x')dx'$$

Solution to homogeneous problem

General method of constructing Green's functions using homogeneous solutions to Sturm-Liouville equations

Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Two homogeneous solutions

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)g_i(x)=0 \qquad \text{for} \quad i=a,b$$

Let

$$G_\lambda(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$$

Digression:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)g_a(x)=0$$

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)g_b(x)=0$$

Consider the Wronskian:

$$W\equiv-\tau(x)\left(g_a(x)\frac{d}{dx}g_b(x)-g_b(x)\frac{d}{dx}g_a(x)\right)$$

Note that:

$$\frac{dW}{dx}=-\frac{d}{dx}\left(\tau(x)\left(g_a(x)\frac{d}{dx}g_b(x)-g_b(x)\frac{d}{dx}g_a(x)\right)\right)$$

$$=-g_a(x)\frac{d}{dx}\left(\tau(x)\frac{d}{dx}g_b(x)\right)+g_b(x)\frac{d}{dx}\left(\tau(x)\frac{d}{dx}g_a(x)\right)$$

$$=-\left(v(x)-\lambda\sigma(x)\right)g_a(x)g_b(x)+\left(v(x)-\lambda\sigma(x)\right)g_b(x)g_a(x)=0$$

For $\epsilon \to 0$:

$$\int_{x'-\epsilon}^{x'+\epsilon} dx\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\int_{x'-\epsilon}^{x'+\epsilon}dx\,\delta(x-x')$$

$$\int_{x'-\epsilon}^{x'+\epsilon} dx\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}\right)\frac{1}{W}g_a(x_<)g_b(x_>)=1$$

$$-\frac{\tau(x)}{W}\left(\frac{d}{dx}g_a(x_<)g_b(x_>)\right)\Bigg]_{x'-\epsilon}^{x'+\epsilon}=-\frac{\tau(x')}{W}\left(g_a(x')\frac{d}{dx'}g_b(x')-g_b(x')\frac{d}{dx'}g_a(x')\right)$$

$$\Rightarrow W=-\tau(x')\left(g_a(x')\frac{d}{dx'}g_b(x')-g_b(x')\frac{d}{dx'}g_a(x')\right)$$

Note -- W (Wronskian) is constant, since $\frac{dW}{dx'}=0.$

$\Rightarrow$ Useful Green's function construction in one dimension:

$$G_\lambda(x,x')=\frac{1}{W}g_a(x_<)g_b(x_>)$$

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function solution for $x_l \le x \le x_u$:

$$\varphi_\lambda(x)=\varphi_{\lambda 0}(x)+\int_{x_l}^{x_u}G_\lambda(x,x')F(x')dx'$$

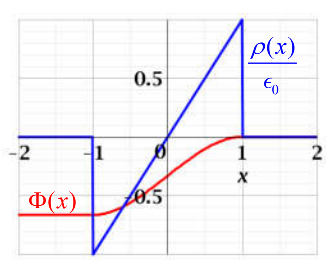
$$=\varphi_{\lambda 0}(x)+\frac{g_b(x)}{W}\int_{x_l}^{x}g_a(x')F(x')dx'+\frac{g_a(x)}{W}\int_{x}^{x_u}g_b(x')F(x')dx'$$

Note that:

$$\varphi_\lambda(x_l)=\varphi_{\lambda 0}(x_l)+\frac{g_a(x_l)}{W}\int_{x_l}^{x_u}g_b(x')F(x')dx'$$

$$\varphi_\lambda(x_u)=\varphi_{\lambda 0}(x_u)+\frac{g_b(x_u)}{W}\int_{x_l}^{x_u}g_a(x')F(x')dx'$$

Example:



$$-\frac{d^2\Phi(x)}{dx^2}=\frac{\rho(x)}{\epsilon_0}$$

$$\rho(x)=\begin{cases}0 & x<-a\\ \rho_0 x/a & -a\le x\le a\\ 0 & x>a\end{cases}$$

Homogeneous solutions

$$\frac{d^2 g_a(x)}{dx^2} = 0 \qquad g_a(x) = 1$$

$$\frac{d^2 g_b(x)}{dx^2} = 0 \qquad g_b(x) = x$$

$$W = -g_a(x)\frac{dg_b(x)}{dx} + g_b(x)\frac{dg_a(x)}{dx} = -1$$

Green's function for this case:

$$G_\lambda(x,x') = \frac{1}{W} g_a(x_<) g_b(x_>) = -x_>$$

Green's function solution for this example

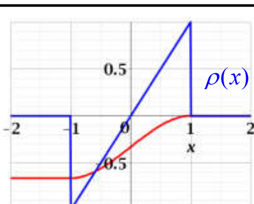
$$\varphi_\lambda(x) = \frac{g_b(x)}{W}\int_{x_l}^{x} g_a(x')F(x')dx' + \frac{g_a(x)}{W}\int_{x}^{x_u} g_b(x')F(x')dx'$$

$$\Rightarrow \Phi(x) = -\frac{x}{\epsilon_0}\int_{-\infty}^{x}\rho(x')dx' - \frac{1}{\epsilon_0}\int_{x}^{\infty} x'\rho(x')dx'$$

$$= \frac{1}{\epsilon_0}\int_{-\infty}^{x}\rho(x')(x'-x)dx' - \frac{1}{\epsilon_0}\int_{-\infty}^{\infty} x'\rho(x')dx'$$

For $x < -a$: $\quad \Phi(x) = -\frac{\rho_0}{a\epsilon_0}\int_{-a}^{a} x'^2 dx' = -\frac{2\rho_0 a^2}{3\epsilon_0}$

For $x > a$: $\quad \Phi(x) = 0$



$$\rho(x) = \begin{cases} 0 & x < -a \\ \rho_0 x/a & -a \le x \le a \\ 0 & x > a \end{cases}$$

$$\Phi(x) = \begin{cases} -\frac{2\rho_0 a^2}{3\epsilon_0} & x < -a \\ \frac{\rho_0}{a\epsilon_0}\left(\frac{1}{3}(x^3 - a^3) - \frac{1}{2}(x^3 - xa^2)\right) & -a \le x \le a \\ 0 & x > a \end{cases}$$

Eigenfunction solution methods of Sturm-Liouville equations (assume all functions and constants are real):

Homogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi_0(x)=0$

Inhomogenous problem: $\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\phi(x)=F(x)$

Eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)\right)f_n(x)=\lambda_n\sigma(x)f_n(x)$$

It is possible to prove several properties of the eigenfunctions $f_n(x)$ – *including* orthogonality and completeness.

Orthogonality statement: $\int_a^b \sigma(x)f_n(x)f_m(x)dx=\delta_{nm}N_n$,

where $N_n \equiv \int_a^b \sigma(x)(f_n(x))^2dx$.

Completeness of eigenfunctions:

$$\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n}=\delta(x-x')$$

Suppose that we can find a Green's function defined as follows:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

In terms of eigenfunctions:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\sigma(x)\sum_n \frac{f_n(x)f_n(x')}{N_n}$$

$$\Rightarrow G_\lambda(x,x')=\sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}$$

Solution to inhomogeneous problem by using Green's functions

Inhomogenous problem:

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)\varphi(x)=F(x)$$

Green's function :

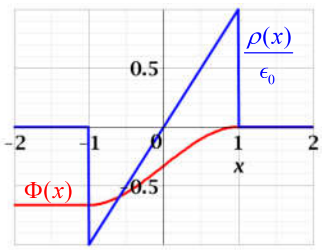
$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_\lambda(x,x')=\delta(x-x')$$

Formal solution:

$$\varphi_\lambda(x)=\varphi_{\lambda 0}(x)+\int_0^L G_\lambda(x,x')F(x')dx'$$

Solution to homogeneous problem

Example:



$$-\frac{d^2\Phi(x)}{dx^2} = \frac{\rho(x)}{\epsilon_0} \qquad \rho(x) = \begin{cases} 0 & x < -a \\ \rho_0 x / a & -a \le x \le a \\ 0 & x > a \end{cases}$$

Eigenvalue equation:

$$\left(-\frac{d^2}{dx^2}\right) f_n(x) = \lambda_n f_n(x)$$

Eigenfunctions

$$f_n(x) = \sqrt{\frac{1}{a}} \sin\left(\frac{n\pi x}{2a}\right)$$

Eigenvalues:

$$\lambda_n = \left(\frac{n\pi}{2a}\right)^2$$

Completeness of eigenfunctions:

$$\sigma(x) \sum_n \frac{f_n(x) f_n(x')}{N_n} = \delta(x - x') \qquad \text{for} \qquad -a \le x \le a$$

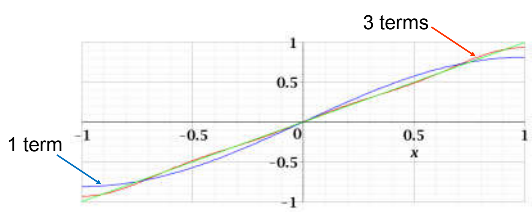
In this example: $$\frac{1}{a} \sum_n \sin\left(\frac{n\pi x}{2a}\right) \sin\left(\frac{n\pi x'}{2a}\right) = \delta(x - x')$$

Digression – convergence of Fourier series representation of ρ(x) itself:

For $-a \le x \le a$:

$$\rho(x) = \frac{\rho_0}{a} x$$

$$\tilde{\rho}(x) = \frac{8\rho_0}{\pi^2}\left(\sin\left(\frac{\pi x}{2}\right) - \frac{1}{9}\sin\left(\frac{3\pi x}{2}\right) + \frac{1}{25}\sin\left(\frac{5\pi x}{2}\right) - \dots\right)$$



Green's function :

$$\left(-\frac{d}{dx}\tau(x)\frac{d}{dx}+v(x)-\lambda\sigma(x)\right)G_{\lambda}(x,x')=\delta(x-x')$$

Green's function for the example:

$$G(x,x')=\sum_n \frac{f_n(x)f_n(x')/N_n}{\lambda_n-\lambda}=\frac{1}{a}\sum_{n\text{ (odd)}}\frac{\sin\left(\frac{n\pi x}{2a}\right)\sin\left(\frac{n\pi x'}{2a}\right)}{\left(\frac{n\pi}{2a}\right)^2}$$

Using Green's function to solve inhomogenous equation:

$$\left(-\frac{d^2}{dx^2}\right)\Phi(x)=\frac{\rho_0}{a\epsilon_0}x \qquad \text{for} \quad -a\le x\le a$$

$$\Phi(x)=\frac{\rho_0}{a\epsilon_0}\int_{-a}^{a}G(x,x')x'dx'$$

$$=\frac{1}{a}\frac{\rho_0}{a\epsilon_0}\sum_{n\text{ (odd)}}\left[\frac{\sin\left(\frac{n\pi x}{2a}\right)}{\left(\frac{n\pi}{2a}\right)^2}\int_{-a}^{a}\sin\left(\frac{n\pi x'}{2a}\right)x'dx'\right]$$

$$\int_{-a}^{a}\sin\left(\frac{n\pi x'}{2a}\right)x'dx'=\frac{8a^2}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) \qquad \text{for } n \text{ odd}$$

$$\Phi(x)=\Phi_0+\frac{32\rho_0a^2}{\pi^4\epsilon_0}\sum_{n\text{ (odd)}}\left[\frac{\sin\left(\frac{n\pi x}{2a}\right)}{n^4}\sin\left(\frac{n\pi}{2}\right)\right]$$